Generalized functions - Exercise 4

Let $n > k$ be integers, and identify of $\mathbb{R}^k$ with the subspace of $\mathbb{R}^n$ with last $n-k$ coordinates being zero. Let W denote the complement $\mathbb{R}^n \setminus \mathbb{R}^k$. Let $C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)$ denote the space of distributions supported on $\mathbb{R}^k$. Recall that we defined

$$V_m(C_c^\infty(\mathbb{R}^n), \mathbb{R}^k) = \{f \in C_c^\infty(\mathbb{R}^n) : \frac{\partial^i f}{(\partial x)^i}_{|\mathbb{R}^k} = 0 \,\forall \text{ index } i \text{ with } |i| \leq m\},$$

and

$$F_m(C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)) = \{\xi \in C^{-\infty}(\mathbb{R}^n) : \xi_{|V_m} = 0\}.$$

1. Show that $\bigcap\limits_{m=0}^{\infty} V_m = \overline{C_c^\infty(W)}$.
2. Show that $\bigcup\limits_{i=0}^{\infty} F_i \neq C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)$.
3. Show that F_m is invariant w.r.t. diffeomorphisms of $\mathbb{R}^n$ preserving $\mathbb{R}^k$.
4. * Let $U \subseteq \mathbb{R}^n$ be open and $\overline{U}$ compact. Show that for every $\xi \in C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)$ there exists $\xi' \in F_m$ such that $\xi_{|U} = \xi'_{|U}$, thus $\bigcup\limits_{m=0}^{\infty} F_m$ covers $C^{-\infty}_{\mathbb{R}^k}(\mathbb{R}^n)$ locally.