Generalized functions - Exercise 5

Solve the following exercises. Questions marked with (*) are optional.

1. Show that for any $x \in \mathbb{Q}$,

$$|x|_\infty \cdot \prod_{p \text{ prime}} |x|_p = 1.$$

2. Let $B_\epsilon(a) = \{x \in \mathbb{Q}_p : |x - a|_p < \epsilon\}$ be the open p-adic ball of radius epsilon around $a \in \mathbb{Q}_p$.

   (a) $B_\epsilon(a)$ is open by definition, Show that it is also closed.

   (b) Show that every point in $B_\epsilon(a)$ is its center.

   (c) Show that there are countable many open balls which contain 0 in $\mathbb{Q}_p$.

3. Let $|\cdot|$ and $|\cdot|'$ be two absolute values on a field F. Show that the following are equivalent:

   (a) $|\cdot|$ and $|\cdot|'$ are equivalent.

   (b) There exists $\alpha \in \mathbb{R}_{>0}$ such that $|\cdot| = (|\cdot|')^\alpha$.

   (c) Every sequence which is Cauchy with respect to $|\cdot|$ is Cauchy with respect to $|\cdot|'$.

4. Let C be the Cantor set.

   (a) Show $\mathbb{Z}_p \cong C$.

   (b) Show $\mathbb{Q}_p \cong C \backslash \{*\}$.

   (c) What are the cardinalities of $\mathbb{Z}_p$ and $\mathbb{Q}_p$? What are the connected components?

   (d) Prove that $\mathbb{Q}_p^n$ and $\mathbb{Q}_p$ are homeomorphic.

   (e) Let $U \subset \mathbb{Q}_p^n$ be an open set. Show that either U is homeomorphic to the Cantor set, or to Cantor set minus a point.

5. (a) Prove Haar's theorem for $(\mathbb{Q}_p, +)$.

   (b) Given a Haar measure μ, we can define another invariant measure $\mu_a(B) = \mu(aB)$ for any $a \in \mathbb{Q}_p$. Show that $\mu_a = |a| \cdot \mu$.