INTRODUCTION TO GENERALIZED FUNCTIONS

Motivation.

Example 1. *δ_t - the Dirac delta function on $\mathbb{R}$ at point t.*
Can be intuitively described by

$$\delta_t(x) := \begin{cases} \infty & x = t \\ 0 & x \neq t \end{cases}, \quad \text{and} \int_{-\infty}^{\infty} \delta_t(x)dx = 1.$$

Note that it also satisfies:

$$\int_{-\infty}^{\infty} \delta_t(x)f(x)dx = f(t) \int_{-\infty}^{\infty} \delta_t(x)dx = f(t).$$

for any continuous function $f(t)$.

Thus, while it is not well-defined as a function, it is well-defined as a functional on the space of continuous functions.

Definition 2. *A* generalized function *or a* distribution *is a continuous functional on the space $C_c^\infty(\mathbb{R})$ of smooth compactly supported functions.*

Continuous w.r.t. the topology of uniform convergence on compacta with all derivatives.

Motivations for generalized functions:

- Every real function $f : \mathbb{R} \to \mathbb{R}$ can be constructed as an (ill-defined) sum of continuum indicator functions $f := \sum_{t \in \mathbb{R}} f(t)\delta_t$.
- In general, solutions to differential equations, and even just derivatives of functions are not functions, but rather generalized

Date: October 26, 2020.

function. Using the language of generalized functions allows one to rigorize such notions.

- Generalized functions are extremely useful in physics. For example, the density of a point mass can be described by the Dirac delta function.

Example 3. *Every locally integrable function f defines a generalized function by*

$$\langle f, g\rangle := \int_{\mathbb{R}} f(t)g(t)dt$$

Definition 4. *A sequence $\phi_n \in C_c(\mathbb{R})$ of continuous, non-negative, compactly supported functions is said to be a δ-sequence if:*

(i) ϕ_n satisfy $\int\limits_{-\infty}^{\infty} \phi_n(x) \cdot dx = 1$ (that is have total mass 1), and

(ii) for any fixed $\varepsilon > 0$, the functions ϕ_n are supported on $[-\varepsilon, \varepsilon]$ for n sufficiently large.

The reason for the name is that any such sequence converges to δ_0 in the space of generalized functions.
Such sequences can be generated, for example, by starting with a non-negative, continuous, compactly supported function ϕ_1 of total mass 1, and by then setting $\phi_n(x) = n\phi_1(nx)$.

Denote $C^{-\infty}(\mathbb{R})$:= the space of all generalized functions.

For any open $U \subset \mathbb{R}$ define $C^{-\infty}(\mathbb{R})$ in the same way.

We have $C_c^{\infty}(U) \subset C_c^{\infty}(\mathbb{R})$ - continuation by zero.
The dual map is restriction $C^{-\infty}(\mathbb{R}) \to C^{-\infty}(U)$.

A third way to define δ_0 is to derive a step function.

Derivatives of generalized functions.
Let $f \in C_c^\infty(\mathbb{R})$. Since

$$\xi_{f'}(\phi) = \int_{-\infty}^{\infty} f'(x) \cdot \phi(x) dx,$$

integration by parts implies

$$\xi_{f'}(\phi) = f(x) \cdot \phi(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) \cdot \phi'(x) dx = \int_{-\infty}^{\infty} f(x) \cdot \phi'(x) dx$$

Thus, we *define*

$$\langle \xi', \phi \rangle := -\langle \xi, \phi' \rangle.$$

Example 5. $\langle \delta_0', f \rangle = f'(0)$.

The support of generalized functions.
For a function f, $\text{supp}(f) = \overline{\{x \in \mathbb{R} \,|\, f(x) \neq 0\}}$.
Not defined for generalized functions, since we cannot evaluate them at points.
Instead, we take $\text{supp}\xi$ to be the minimal closed subset $X \subset \mathbb{R}$ such that the restriction of ξ to the complement X^c of X is 0.

Example 6. *The support of* δ_0 *is* $\{0\}$.

Remark 7. *Warning! While the support of* δ_0' *is also* $\{0\}$*, given some* $f \in C^\infty(\mathbb{R})$ *for which* $f(0) = 0$ *but* $f'(0) \neq 0$*, we get that* $\delta_0'(f) = -\delta_0(f') = -f'(0) \neq 0$*. In other words, having* $f(0) = 0$ *is not enough to get* $\langle \delta_0', f \rangle = 0$*, we need* $f'(0)$ *to vanish.*

Exercise 8. *Let* $\xi_1, \xi_2 \in C^{-\infty}(\mathbb{R})$ *and* $a, b \in \mathbb{R}$*. Show that:*

(1) $\text{supp}(a\xi_1 + b\xi_2) \subseteq \text{supp}(\xi_1) \cup \text{supp}(\xi_2)$.
(2) $\text{supp}(\xi) - \text{supp}(\xi)^\circ \subseteq \text{supp}(\xi') \subseteq \text{supp}(\xi)$.

Remark 9. *All the generalized functions* $\xi \in C_c^{-\infty}(\mathbb{R})$ *which are supported on* $\{0\}$ *are of the form* $\sum_{i=0}^{n} c_i \delta^{(i)}$ *for some* $n \in \mathbb{N}_0$ *and* $c_i \in \mathbb{R}$.

Products and convolutions of generalized functions. For $\phi \in C_c^{\infty}(\mathbb{R})$ and $f, g \in C(\mathbb{R})$ we have

$$\langle f \cdot g, \phi \rangle = \int_{-\infty}^{\infty} \xi(x) \cdot f(x) \cdot \phi(x) dx = \langle g, f \cdot \phi \rangle$$

Definition 10. *Let* $f \in C(\mathbb{R})$ *and* $\xi \in C_c^{-\infty}(\mathbb{R})$. *Define* $f \cdot \xi$ *by*

$$\langle f \cdot \xi)(\phi\rangle := \langle \xi, f \cdot \phi \rangle$$

The product of two generalized functions is not always defined.

Given two functions f, g, their *convolution* is defined by

$$(f * g)(x) := \int_{-\infty}^{\infty} f(t) \cdot g(x - t) dt.$$

The convolution of two smooth functions is always smooth, if it exists. In addition, if f and g have compact support, then so does $f * g$:

Exercise 11. *Show that* $\mathrm{supp}(f * g) \subseteq \overline{\mathrm{supp}(f) + \mathrm{supp}(g)}$, *where* $\mathrm{supp}(f) + \mathrm{supp}(g)$ *is the* Minkowski sum *of* $\mathrm{supp}(f)$ *and* $\mathrm{supp}(g)$.

Given $f, g \in C_c^{\infty}(\mathbb{R})$ we can write $(f * g)(x) = \langle \xi_f, \tilde{g}_x) \rangle$,
where $\tilde{g}_x(t) := g(x - t)$.
This motivates us to define the convolution $\xi * g$ as the function

$$(\xi * g)(x) = \xi(\tilde{g}_x).$$

Exercise 12. *Show that for* $\phi \in C_c^{\infty}(\mathbb{R})$ *and* $\xi \in C^{-\infty}(\mathbb{R})$ *we get that* $\xi * \phi$ *is a smooth function.*

One can also define the convolution of two generalized functions if one of them is compactly supported.

Exercise 13. *Let* $\xi \in C^{-\infty}(\mathbb{R})$. *In an exercise above we showed: if* $\phi \in C_c^{\infty}(\mathbb{R})$ *then the convolution* $\xi * \phi$ *is smooth. Show that if* $\mathrm{supp}(\xi)$ *is compact, then* $\xi * \phi$ *is still smooth.*

To summarize, the convolution of two compactly supported distributions is well defined and compactly supported, while the convolution of a compactly supported distribution with an arbitrary distribution is well defined, but usually not compactly supported.

Exercise 14. *Show the following identities for any*

$$\phi, \psi \in C_c^\infty \text{ and } \xi \in C^{-\infty}(\mathbb{R}).$$

(i) $\delta_0 * \phi = \phi$.
(ii) $\delta_0' * \phi = \phi'$.
(iii) $\phi * \psi = \phi * \psi$.
(iv) $\phi * (\psi * \xi) = (\phi * \psi) * \xi$.
(v) $(\phi * \xi)' = \phi * \xi' = \phi' * \xi$.

Generalized functions on $\mathbb{R}^n$. All the notions above make sense for functions and generalized functions in several variables. The definitions and the statements literally generalize to this case.

Generalized functions and differential operators.
Let A be a differential operator with constant coefficients,
e.g. $Af := f'' + 5f' + 6f$.

We want to solve the differential equation $Af = f_0$, where f_0 is some fixed function. Green's idea:

First solve the equation $AG = \delta_0$.
Then, $f := G * f_0$ is the solution to $Af = f_0$.
Indeed, $f' = G' * f_0$, thus

$$Af = AG * f_0 = \delta_0 * f_0 = f_0.$$

Hence, we can find a general solution f for $Af = g$ by solving a single simpler equation $AG = \delta_0$.
The solution G is called *Green's function of the operator*.

Exercise 15. *Solve the equation* $\Delta G = \delta_0$ *(where* $\Delta = \frac{\partial^2}{\partial x^2}$ *is the 1-Laplacian).*

Regularization of generalized functions.

Definition 16. *Let* $\{\xi_\lambda\}_{\lambda\in\mathbb{C}}$ *be a family of generalized functions. We say the family is* analytic *if* $\langle \xi_\lambda, f\rangle$ *is analytic as a function of* $\lambda \in \mathbb{C}$ *for every* $f \in C_c^\infty(\mathbb{R})$.

Example 17. *We denote*

$$x_+^\lambda := \begin{cases} x^\lambda & x > 0 \\ 0 & x \le 0 \end{cases},$$

and define the family by $\xi_\lambda := x_+^\lambda$ $Re(\lambda) > -1$. *The behavior of the function changes as* λ *changes: When* $Re(\lambda) > 0$ *we have a continuous function; if* $Re(\lambda) = 0$ *we get a step function and for* $Re(\lambda) \in (-1, 0)$, x_+^λ *will not be bounded. We would like to extend the definition analytically to* $Re(\lambda) < -1$.
Deriving x_+^λ *(both as a complex function or as we defined for generalized function) gives* $\xi'_\lambda = \lambda \cdot \xi_{\lambda-1}$. *This is a functional equation which enables us to* ***define*** $\xi_{\lambda-1} := \frac{\xi'_\lambda}{\lambda}$, *and thus extend* ξ_λ *to every* $\lambda \in \mathbb{C}$ *such that* $Re(\lambda) > -2$, *and for every* λ *by reiterating this process. This extension is not analytic, but it is meromorphic: it has a pole in* $\lambda = 0$, *and by the extension formula, in* $\lambda = -1, -2, \ldots$.

This is an example of *a meromorphic family of generalized functions*. We now give a formal definition. The family $\{\xi_\lambda\}_{\lambda\in\mathbb{C}}$ has a set of poles $\{\lambda_n\}$ (poles are always discrete), whose respective orders are denoted $\{d_n\}$. A family of generalized functions is called *meromorphic* if every pole λ_i has a neighborhood U_i, such that $\langle \xi_\lambda, f\rangle$ is analytic for every $f \in C_c^\infty(\mathbb{R})$ and $\lambda_i \neq \lambda \in U_i$.

Exercise 18. *Find the order and the leading coefficient of every pole of* $\xi_\lambda := x_+^\lambda$.

Example 19. *For a given* $p \in \mathbb{C}[x_1, \ldots x_n]$, *similarly to before set,*

$$p_+(x_1, \ldots x_n)^\lambda := \begin{cases} p(x_1, \ldots x_n)^\lambda & x > 0 \\ 0 & x \le 0 \end{cases}.$$

The problem of finding the meromorphic continuation of a general polynomial was open for some time. It was solved by J. Bernstein by proving that there exists a differential operator $Dp_+^\lambda := b(\lambda) \cdot p_+^{\lambda-1}$, *where* $b(\lambda)$ *is a polynomial pointing on the location of the poles.*

Exercise 20. *Solve the problem of finding an analytic continuation for* $p_+(x_1, \dots x_n)^\lambda$ *in the following cases:*

(1) $p(x, y, z) := x^2 + y^2 + z^2 - a$ *and* $a \in \mathbb{R}$.
(2) $p(x, y, z) := x^2 + y^2 - z^2$.