INVERTIBLE TOP FORM ON THE HILBERT SCHEME OF A PLANE IN POSITIVE CHARACTERISTIC

AVRAHAM AIZENBUD, DMITRY GOUREVITCH, DAVID KAZHDAN, AND EITAN SAYAG

ABSTRACT. We prove that the Hilbert scheme of the plane in positive characteristic admits an invertible top differential form.

This implies certain integrability properties of the symmetric powers of the plane. This allows to define a function on the collection of monic polynomials over a local field which can be thought of as a variant of the inverse square root of the discriminant. In characteristic 0 it essentially coincides with this inverse square root, however in general it is quite different, and unlike this inverse square root, it is locally summable. In a sequel work [AGKS] we use this local summability in order to prove the positive characteristic analog of Harish-Chandra's local integrability theorem of characters of representations under certain conditions.

The main results of this paper are known in characteristic zero. In fact a stronger result is known: there is a symplectic form on the Hilbert scheme of a plane.

CONTENTS

Date: February 17, 2026.

2020 *Mathematics Subject Classification.* 14C05, 14B05, 14E15.

Key words and phrases. Hilbert scheme, Hilbert-Chow map, positive characteristic, singularity, symmetric power, discriminant.

1. Introduction

Throughout the paper we fix a field F of arbitrary characteristic. We will also fix a natural number n.

1.1. **The Hilbert Scheme.** In order to formulate our results let us first recall the definition of the Hilbert scheme.

Definition 1.1.1. *Let Sch_F be the category of F-Schemes. For an F-algebraic variety $\mathbf{Z}$ define the Hilbert functor $Hilb_n(\mathbf{Z}) : Sch_F^{op} \to sets$ by*

$$Hilb_n(\mathbf{Z})(\mathbf{S}) := \{\text{sub-scheme } \mathbf{Y} \subset \mathbf{S} \times \mathbf{Z} \,|\, (pr_{\mathbf{S}})_*(\mathcal{O}_{\mathbf{Y}}) \text{ is locally free of rank } n \text{ over } \mathbf{S}, \}$$

where $pr_{\mathbf{S}}$ is the projection.

Theorem 1.1.2 ([Gro62], see also [BK05, Theorem 7.2.3])**.** *If $\mathbf{Z}$ is a quasi-projective variety then the Hilbert functor $Hilb_n(\mathbf{Z})$ is representable by a scheme which we denote by $\mathbf{Z}^{[n]}$.*

Theorem 1.1.3 (See *e.g.* [BK05, Theorem 7.4.1])**.** *If $\mathbf{Z}$ is a smooth quasi-projective irreducible algebraic surface then $\mathbf{Z}^{[n]}$ is a smooth irreducible variety of dimension $2n$.*

1.2. **Main results.** We prove the following:

Theorem A. *There exists an invertible top differential form on $(\mathbb{A}^2)^{[n]}$.*

1.3. **Relation to the singularities of the symmetric power of the plane.** Theorem A is related to the singularities of the symmetric power of the plane. In order to formulate this relation we introduce some notations:

Definition 1.3.1. *For a quasi projective algebraic variety $\mathbf{Z}$ define its symmetric power by:*

$$\mathbf{Z}^{(n)} := \mathbf{Z}^n // S_n.$$

Here $//$ denotes the categorical quotient. By Corollary 3.1.8 below this quotient exists.

Notation 1.3.2. *Let $\mathbf{Z}$ be a quasi-projective variety. Let $x \in \mathbf{Z}^{[n]}(\bar{F})$. It corresponds to a sheaf of ideals $\mathcal{I}_x \subset \mathcal{O}_{\mathbf{Z}_{\bar{F}}}$. For any $z \in \mathbf{Z}(\bar{F})$ denote*

$$n_x(z) := \dim(\mathcal{O}_{\mathbf{Z}_{\bar{F}},z}/(\mathcal{I}_x)_z).$$

This gives a multiset in $\mathbf{Z}(\bar{F})$ of size n. By Lemma 3.0.2 below, we can interpret this multiset as a point in $\mathbf{Z}^{(n)}(\bar{F})$. Denote this point by $\mathfrak{H}_{\mathbf{Z},n}(x)$.

Theorem 1.3.3 ([Ive70, II.2,II.3], [BK05, Theorem 7.3.1][1])**.** *Let* $\mathbf{Z}$ *be a quasi-projective variety. There exists (and unique) a projective morphism* $\mathfrak{H}_{\mathbf{Z},n} : \mathbf{Z}^{[n]} \to \mathbf{Z}^{(n)}$ *that gives on the level of* $\bar{F}$ *points the map* $\mathfrak{H}_{\mathbf{Z},n}$ *defined above.*

This morphism is called the Hilbert-Chow morphism.

Theorems 1.1.3 and 1.3.3 imply:

Corollary 1.3.4. *Let* $\mathbf{Z}$ *be a (quasi-projective) smooth surface. Then the Hilbert-Chow map* $\mathfrak{H}_{\mathbf{Z},n} : \mathbf{Z}^{[n]} \to \mathbf{Z}^{(n)}$ *is a resolution of singularities.*

Theorem A is related to the properties of this resolution. In order to formulate these relations we make:

Definition 1.3.5.

(i) We recall that a modification $\gamma : \tilde{\mathbf{V}} \to \mathbf{V}$ *of algebraic varieties is a birational proper morphism.*
(ii) We call a modification $\gamma : \tilde{\mathbf{V}} \to \mathbf{V}$ *of algebraic varieties integrable if for any open* $\mathbf{U} \subset \mathbf{V}$ *and any top-form* ω *on the smooth locus of* $\mathbf{U}$ *the rational form* $\gamma^*(\omega)$ *on* $\gamma^{-1}(\mathbf{U})$ *is regular on the smooth locus of* $\gamma^{-1}(\mathbf{U})$*.*
(iii) We call such modification sharply integrable if $\gamma^*(\omega)$ *vanishes only on* $\gamma^{-1}(\bar{\mathbf{D}})$ *where* $\mathbf{D}$ *is the zero locus of* ω*.*
(iv) We call a variety (sharply) integrable if it admits a (sharply) integrable resolution of singularities.

Remark 1.3.6. *In characteristic zero, one can show that TFAE:*[2]

(a) the singularities of $\mathbf{Z}$ *are rational,*
(b) $\mathbf{Z}$ *is integrable and Cohen-Macaulay.*

In positive characteristic there is no single accepted definition of rational singularities, and one can take condition (b) as a definition.

We will see that Theorem A follows from:

Theorem B. *The Hilbert-Chow map*

$$\mathfrak{H}_{\mathbb{A}^2,n} : (\mathbb{A}^2)^{[n]} \to (\mathbb{A}^2)^{(n)}$$

is a sharply integrable modification.

This implies:

Corollary C. $(\mathbb{A}^2)^{(n)}$ *is sharply integrable.*

Remark 1.3.7. *In fact, it is easy to see that Theorem B and Theorem A are equivalent. So one could instead prove directly Theorem A and deduce Theorem B.*

[1] The theorem is formulated in [BK05] for algebraically closed fields. However, it is based on results from [Ive70] which do not make this assumption, so the proof is valid for any field.

[2] See e.g. [AA16, Appendix B, Proposition 6.2])

1.4. **Background and motivation.**

1.4.1. *The characteristic zero case.* The characteristic zero counterpart of the main results of this paper is well known. In fact, stronger results are known. Namely, the Hilbert-Chow map for smooth surfaces in characteristic 0 is a symplectic resolution (see *e.g.* [Nak99, Theorem 1.17]). This implies the characteristic zero counterpart of Theorem B. This also implies that in characteristic zero the Hilbert scheme of the plane is a symplectic variety. This in turn implies the characteristic 0 counterparts of Theorem A and Corollary C. In addition, $(\mathbb{A}^2)^{(n)}$ is a quotient of algebraic variety by a finite group, therefore, by [Bou87, Corollaire], in characteristic zero its singularities are rational. As mentioned in Remark 1.3.6 this is equivalent to the fact that it is integrable and Cohen-Macaulay.

1.4.2. *Relation to local finiteness of measures.* If the field F is local, integrability of an algebraic variety $\mathbf{Z}$ implies that given a top form ω on its smooth locus, the corresponding measure $|\omega|$ on $\mathbf{Z}(F)$ is locally finite.

Therefore, given a (locally) dominant map $\phi : \mathbf{Z} \to \mathbf{Y}$ to a smooth variety and a function $f \in C_c^\infty(\mathbf{Z}(F))$, the measure $\phi_*(f|\omega|)$ is also locally finite. Since it is also absolutely continuous (w.r.t. a smooth invertible measure on $\mathbf{Y}(F)$), this measure has a locally summable density function.

Applying this consideration to the map $(\mathbb{A}^2)^{(n)} \to (\mathbb{A}^1)^{(n)}$ (induced by the projection $\mathbb{A}^2 \to \mathbb{A}^1$) we get a locally summable density function η (defined up to multiplication by a smooth compactly supported function) on $(\mathbb{A}^1)^{(n)}(F)$. Note that $(\mathbb{A}^1)^{(n)}$ is naturally identified with the space of monic polynomials of degree n.

Over $\mathbb{C}$ this function is the absolute value of the inverse square root of the discriminant – $|\Delta|^{-\frac{1}{2}}$. Over a general local field of characteristic zero, this function is bounded from above and from below by a constant times $|\Delta|^{-\frac{1}{2}}$. However, in positive characteristic this is no longer true. Moreover, in small positive characteristic the function $|\Delta|^{-\frac{1}{2}}$ is not locally summable. One can consider η as a better behaved version of $|\Delta|^{-\frac{1}{2}}$. In a sequel work [AGKS] we use the local summability of η in order to prove the positive characteristic analog of Harish-Chandra's integrability theorem under certain conditions. It turns out that for the sake of this theorem η actually plays the role of $|\Delta|^{-\frac{1}{2}}$.

1.5. **Idea of the proof.** We define a closed subset

$$(\mathbb{A}^2)^{(n)}_{diag} \subset (\mathbb{A}^2)^{(n)}$$

which corresponds to the diagonal copy of $\mathbb{A}^2$. We prove:

Lemma D (For a precise formulation see Lemma 4.0.6)**.** *Outside $(\mathbb{A}^2)^{(n)}_{diag}$ the Hilbert-Chow map looks (locally in the étale topology) like a product of the Hilbert-Chow maps for smaller values of n.*

We use this Lemma together with the induction hypothesis in order to prove the theorem outside $(\mathbb{A}^2)^{(n)}_{diag}$. Then we use the fact that the complement to

$$\mathfrak{H}^{-1}_{\mathbb{A}^2,n}((\mathbb{A}^2)^{(n)}_{diag})$$

in $(\mathbb{A}^2)^{[n]}$ is big in order to deduce the result.

This strategy works only for $n > 2$, for $n = 2$ we prove the theorem by an explicit computation.

1.6. **Structure of the paper.** In §2 we fix conventions that will be used throughout the paper.

In §3 we study quotients of varieties by finite group actions. In §4 we prove Theorem B. In §4.1 we prove Lemma D.

In §5 we prove Theorem A.

1.7. **Acknowledgments.** We wish to thank Pavel Etigof and Victor Ginzburg for discussing with us the situation with Hilbert scheme in positive characteristic.

During the preparation of this paper, A.A., D.G. and E.S. were partially supported by the ISF grant no. 1781/23. D.K. was partially supported by an ERC grant 101142781.

2. Conventions

(a) By a variety we mean a reduced scheme of finite type over F.
(b) When we consider a fiber product of varieties, we always consider it in the category of schemes. We use set-theoretical notations to define subschemes, whenever no ambiguity is possible.
(c) We will usually denote algebraic varieties by bold face letters (such as $\mathbf{X}$) and the spaces of their F-points by the corresponding usual face letters (such as $X := \mathbf{X}(F)$). We use the same conventions when we want to interpret vector spaces as algebraic varieties.
(d) We will use the same letter to denote a morphism between algebraic varieties and the corresponding map between the sets of their F-points.
(e) We will use the symbol $\square$ in a middle of a square diagram in order to indicate that the square is Cartesian.
(f) We will use numbers in a middle of a square diagram in order to refer to the square by the corresponding number.
(g) A big open set of an algebraic variety $\mathbf{Z}$ is an open set whose complement is of co-dimension at least 2 (in each component).
(h) For a variety $\mathbf{Z}$ we denote its smooth locus by $\mathbf{Z}^{sm}$.
(i) For a smooth variety $\mathbf{Z}$ we denote by $\Omega^{top}(\mathbf{Z})$ the sheaf of top differential forms on $\mathbf{Z}$.
(j) For a variety $\mathbf{Z}$ and a field extension E/F, denote by $\mathbf{Z}_E$ the extension of scalars to E. We use similar notation for morphisms.

3. Factorizable actions

In this section we give some standard facts about quotients of an algebraic variety by a finite group which are slightly less standard in positive characteristic.

Definition 3.0.1. *Let a finite group* Γ *act on a variety* $\mathbf{Z}$*. We say that this action is factorizable if the categorical quotient* $\mathbf{Z}//\Gamma$ *exists (as a variety), and the map* $\mathbf{Z} \to \mathbf{Z}//\Gamma$ *is finite.*

Lemma 3.0.2. *Let a finite group* Γ *act factorizably on a variety* $\mathbf{Z}$*. Then the map* $\gamma : \mathbf{Z}(\bar{F})/\Gamma \to (\mathbf{Z}//\Gamma)(\bar{F})$ *is a bijection, where* $\mathbf{Z}(\bar{F})/\Gamma$ *denotes the set of* Γ*-orbits in* $\mathbf{Z}(\bar{F})$*.*

Proof. Since the map is affine, we can assume that $\mathbf{Z}$ is affine. The map γ is onto by the going up theorem. To show that it is one-to-one it is enough to show that for every $O_1, O_2 \in \mathbf{Z}(\bar{F})/\Gamma$ there exists $f \in (\bar{F}[\mathbf{Z}])^{\Gamma}$ such that $f|_{O_1} = 0$ and $f|_{O_2} = 1$. Let $f' \in \bar{F}[\mathbf{Z}]$ such that $f|_{O_1} = 0$ and $f|_{O_2} = 1$, and let $f := \prod_{g \in \Gamma} g^*(f')$. $\square$

3.1. **Factorizabilty of quasi-projective varieties and compatibility with open embeddings.**

In this subsection we prove that quasi-projective varieties are factorizable (see Corollary 3.1.8) and the quotients of factorizable varieties are compatible with open embeddings (see Corollary 3.1.5).

The only place where the positivity of characteristic presents an additional difficulty is the compatibility for the affine case. See the base of the induction in the proof of Lemma 3.1.3. There we can not use the standard averaging method, and we use its multiplicative version instead.

Proposition 3.1.1 (cf. [Har95, Lec. 10, pp. 124-125] or [Ser84])**.** *Let a finite group* Γ *act on an affine variety* $\mathbf{Z}$*. Then the action is factorizable and* $\mathbf{Z}//\Gamma \cong \operatorname{Spec}(\mathcal{O}(\mathbf{Z})^{\Gamma})$*.*

Lemma 3.1.2. *Let a finite group* Γ *act on a variety* $\mathbf{Z}$*. Let* $\mathbf{Z} = \mathbf{Z}_1 \cup \mathbf{Z}_2$ *be a cover of* $\mathbf{Z}$ *by open,* Γ*-invariant factorizable sets. Assume that* $\mathbf{V} := \mathbf{Z}_1 \cap \mathbf{Z}_2$ *is also factorizable and we have Cartesian squares:*

$$(1)\qquad \begin{array}{ccc} \mathbf{V} & \longrightarrow & \mathbf{Z}_1 \\ \downarrow & \square & \downarrow \\ \mathbf{V}//\Gamma & \longrightarrow & \mathbf{Z}_1//\Gamma \end{array} \qquad \begin{array}{ccc} \mathbf{V} & \longrightarrow & \mathbf{Z}_2 \\ \downarrow & \square & \downarrow \\ \mathbf{V}//\Gamma & \longrightarrow & \mathbf{Z}_2//\Gamma \end{array}$$

with the (lower) horizontal maps being open embeddings. Let

$$\mathbf{W} := \mathbf{Z}_1//\Gamma \sqcup_{\mathbf{V}//\Gamma} \mathbf{Z}_2//\Gamma.$$

Then the natural map $\mathbf{Z} \to \mathbf{W}$ *is the categorical quotient map and it is finite. Moreover, we have the following Cartesian squares:*

$$\begin{array}{ccc} \mathbf{Z}_1 & \longrightarrow & \mathbf{Z} \\ \downarrow & \square & \downarrow \\ \mathbf{Z}_1//\Gamma & \longrightarrow & \mathbf{Z}//\Gamma \end{array} \qquad \begin{array}{ccc} \mathbf{Z}_2 & \longrightarrow & \mathbf{Z} \\ \downarrow & \square & \downarrow \\ \mathbf{Z}_2//\Gamma & \longrightarrow & \mathbf{Z}//\Gamma \end{array} \tag{2}$$

Proof. Let $\gamma : \mathbf{Z} \to \mathbf{A}$ be a Γ-invariant map to an algebraic variety. The maps $\gamma|_{\mathbf{Z}_1}$, $\gamma|_{\mathbf{Z}_2}$, and $\gamma|_{\mathbf{V}}$ factor through maps $\alpha : \mathbf{Z}_2//\Gamma \to \mathbf{A}$, $\beta : \mathbf{Z}_1//\Gamma \to \mathbf{A}$ and $\delta : \mathbf{V}//\Gamma \to \mathbf{A}$. These maps give a factorization of γ via a map $\mathbf{W} \to \mathbf{A}$. The uniqueness of such factorization is proven similarly (but simpler). Thus we have proven that the natural map $\mathbf{Z} \to \mathbf{W}$ is the categorical quotient.

Let us now show that the diagrams (2) are Cartesian. Let $\phi_\Gamma : \mathbf{Z} \to \mathbf{Z}//\Gamma \cong \mathbf{W}$ be the categorical quotient map. We need to show that $\phi_\Gamma^{-1}(\mathbf{Z}_1//\Gamma) = \mathbf{Z}_1$ (and similarly for $\mathbf{Z}_2$). Let $x \in \phi_\Gamma^{-1}(\mathbf{Z}_1//\Gamma)$. If $x \in \mathbf{Z}_1$ we are done. Otherwise $x \in \mathbf{Z}_2$. This implies that $\phi_\Gamma(x) \in \mathbf{V}//\Gamma$. Thus, by the right Cartesian square in (1) we get that $x \in \mathbf{V}$ and we are done.

Finally, the finiteness of ϕ_Γ follows from the Cartesian squares (2) and the finiteness of the maps $\mathbf{Z}_i \to \mathbf{Z}_i//\Gamma$. $\square$

Lemma 3.1.3. *Let a finite group* Γ *act on a variety* $\mathbf{Z}$*. Let* $\mathbf{U} \subset \mathbf{Z}$ *be an open* Γ*-invariant set. Assume that* $\mathbf{Z}$ *can be covered by open affine* Γ*-invariant sets. Then*

(i) the action of Γ *on* $\mathbf{Z}$ *is factorizable.*
(ii) The action of Γ *on* $\mathbf{U}$ *is factorizable.*
(iii) The following natural diagram is a Cartesian square.

$$\begin{array}{ccc} \mathbf{U} & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{U}//\Gamma & \longrightarrow & \mathbf{Z}//\Gamma \end{array}$$

(iv) The bottom arrow in the diagram is an open embedding.

Proof. We prove the statement by induction on the size N of the (minimal) cover of $\mathbf{Z}$ by open affine Γ-invariant sets.

Base $N = 1$: (i) follows from Proposition 3.1.1. This implies also that $\mathbf{Z}//\Gamma$ is affine.

Let $\mathbf{A} \subset \mathbf{Z}$ be the complement of $\mathbf{U}$. For any closed point $x \in \mathbf{U}$, we can find a function $f_x \in \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})$ s.t. $f_x(\mathbf{A}) = 0$ and $f_x(\Gamma \cdot x) = \{1\}$.

Let

$$g_x = \prod_{\gamma \in \Gamma} \gamma^*(f_x).$$

Let $\mathbf{U}_x \subset \mathbf{Z}$ be the non-vanishing locus of g_x. Note that $g_x \in \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})^\Gamma = \mathcal{O}_{\mathbf{Z}//\Gamma}(\mathbf{Z}//\Gamma)$. Let $\mathbf{V}_x \subset \mathbf{Z}//\Gamma$ be the non-vanishing locus of g_x when considered as a function on $\mathbf{Z}//\Gamma$. By Proposition 3.1.1, the action of Γ on $\mathbf{U}_x$ is factorizable. $\mathbf{U}_x//\Gamma \cong \mathbf{V}_x$. Let

$$\mathbf{V} = \bigcup_{x \in \mathbf{Z} \text{ is closed}} \mathbf{V}_x$$

It is easy to deduce that $\mathbf{V} \cong \mathbf{U}//\Gamma$ and we have the required Cartesian square.

Step: Write $\mathbf{Z} = \bigcup_{i=1}^{N} \mathbf{Z}_i$ where $\mathbf{Z}_i$ are open, affine, and Γ-invariant. Let $\mathbf{Y} = \bigcup_{i=2}^{N} \mathbf{Z}_i$. Let $\mathbf{V} := \mathbf{Z}_1 \cap \mathbf{Y}$. The previous lemma (Lemma 3.1.2) and the induction hypothesis applied to the pairs $\mathbf{V} \subset \mathbf{Z}_1$, $\mathbf{V} \subset \mathbf{Y}$, $\mathbf{V} \cap \mathbf{U} \subset \mathbf{Z}_1 \cap \mathbf{U}$, and $\mathbf{V} \cap \mathbf{U} \subset \mathbf{Y} \cap \mathbf{U}$ imply (i) and (ii).

We also get the following Cartesian squares:

$$\begin{array}{ccc} \mathbf{Z}_1 & \longrightarrow & \mathbf{Z} \\ \downarrow & \square & \downarrow \\ \mathbf{Z}_1//\Gamma & \longrightarrow & \mathbf{Z}//\Gamma \end{array} \qquad \begin{array}{ccc} \mathbf{Y} & \longrightarrow & \mathbf{Z} \\ \downarrow & \square & \downarrow \\ \mathbf{Y}//\Gamma & \longrightarrow & \mathbf{Z}//\Gamma \end{array} \tag{3}$$

$$\begin{array}{ccc} \mathbf{Z}_1 \cap \mathbf{U} & \longrightarrow & \mathbf{U} \\ \downarrow & \square & \downarrow \\ (\mathbf{Z}_1 \cap \mathbf{U})//\Gamma & \longrightarrow & \mathbf{U}//\Gamma \end{array} \qquad \begin{array}{ccc} \mathbf{Y} \cap \mathbf{U} & \longrightarrow & \mathbf{U} \\ \downarrow & \square & \downarrow \\ (\mathbf{Y} \cap \mathbf{U})//\Gamma & \longrightarrow & \mathbf{U}//\Gamma \end{array} \tag{4}$$

with horizontal maps being open embeddings. Moreover,

$$\mathbf{Z}//\Gamma = \mathbf{Z}_1//\Gamma \cup \mathbf{Y}//\Gamma \text{ and } \mathbf{U}//\Gamma = (\mathbf{Z}_1 \cap \mathbf{U})//\Gamma \cup (\mathbf{Y} \cap \mathbf{U})//\Gamma \tag{5}$$

Applying the induction hypothesis for the pairs $\mathbf{Z}_1 \cap \mathbf{U} \subset \mathbf{Z}_1$ and $\mathbf{Y} \cap \mathbf{U} \subset \mathbf{Y}$ we obtain the following Cartesian squares:

$$\begin{array}{ccc} \mathbf{Z}_1 \cap \mathbf{U} & \longrightarrow & \mathbf{Z}_1 \\ \downarrow & \square & \downarrow \\ (\mathbf{Z}_1 \cap \mathbf{U})//\Gamma & \longrightarrow & \mathbf{Z}_1//\Gamma \end{array} \qquad \begin{array}{ccc} \mathbf{Y} \cap \mathbf{U} & \longrightarrow & \mathbf{Y} \\ \downarrow & \square & \downarrow \\ (\mathbf{Y} \cap \mathbf{U})//\Gamma & \longrightarrow & \mathbf{Y}//\Gamma \end{array} \tag{6}$$

with horizontal maps being open embeddings.

This together with (3) gives the following Cartesian squares:

$$\begin{array}{ccc} \mathbf{Z}_1 \cap \mathbf{U} & \longrightarrow & \mathbf{Z} \\ \downarrow & \square & \downarrow \\ (\mathbf{Z}_1 \cap \mathbf{U})//\Gamma & \longrightarrow & \mathbf{Z}//\Gamma \end{array} \qquad \begin{array}{ccc} \mathbf{Y} \cap \mathbf{U} & \longrightarrow & \mathbf{Z} \\ \downarrow & \square & \downarrow \\ (\mathbf{Y} \cap \mathbf{U})//\Gamma & \longrightarrow & \mathbf{Z}//\Gamma \end{array} \tag{7}$$

with horizontal maps being open embeddings.

This together with (5) proves (iv). It remains to prove (iii). For this it is enough to show that $\phi^{-1}(\mathbf{U}//\Gamma) = \mathbf{U}$, where $\phi : \mathbf{Z} \to \mathbf{Z}//\Gamma$ denotes the quotient map. This follows from (4) and (5).

$\square$

The last Lemma gives us 2 corollaries:

Corollary 3.1.4. *Let a finite group* Γ *act on a variety* $\mathbf{Z}$*. Then TFAE:*

(i) the action of Γ *is factorizable.*
(ii) $\mathbf{Z}$ *can be covered by open affine* Γ*-invariant sets.*

Corollary 3.1.5. *Let a finite group* Γ *act factorizably on a variety* $\mathbf{Z}$*. Let* $\mathbf{U} \subset \mathbf{Z}$ *be an open* Γ*-invariant set. Then*

(i) the action of Γ *on* $\mathbf{U}$ *is factorizable.*
(ii) The following natural diagram is a Cartesian square.

$$\begin{array}{ccc} \mathbf{U} & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{U}//\Gamma & \longrightarrow & \mathbf{Z}//\Gamma \end{array}$$

(iii) The bottom arrow in the diagram is an open embedding.

Lemma 3.1.6. *Let* $\mathbf{Z}$ *be a quasi-projective variety, and* $\mathbf{A} \subset \mathbf{Z}$ *be a finite subvariety. Then there exists an open affine* $\mathbf{V} \subset \mathbf{Z}$ *s.t.* $\mathbf{A} \subset \mathbf{V}$*.*

Proof.

Case 1. $\mathbf{Z}$ is the projective space and F is infinite.
In this case one can take $\mathbf{V}$ to be a compliment to a hyperplane that does not intersect $\mathbf{A}$.

Case 2. $\mathbf{Z}$ is the projective space.[3]
By the previous case we may assume that F is finite (and hence perfect). From the previous case we have an open affine subset $\mathbf{V}' \subset \mathbf{Z}_{\bar{F}}$ that includes $\mathbf{A}_{\bar{F}}$. We can find a finite (Galois) extension E/F s.t. there exists $\mathbf{V}'' \subset \mathbf{Z}_E$ satisfying $\mathbf{V}''_{\bar{F}} = \mathbf{V}'$. Now, we can find $\mathbf{V}$ s.t. $\mathbf{V}_E = \bigcup_{\alpha \in Gal(E/F)} \alpha(\mathbf{V}'')$. It is easy to see that $\mathbf{V}$ satisfies the requirements.

Case 3. $\mathbf{Z}$ is a projective variety.
Follows from the previous case.

Case 4. $\mathbf{Z}$ is a quasi-affine variety.
Embed $\mathbf{Z}$ as an open subset of an affine variety $\mathbf{Z}'$. Let $\mathbf{W}$ be the complement to $\mathbf{Z}$ in $\mathbf{Z}'$. Let $f \in \mathcal{O}_{\mathbf{Z}'}(\mathbf{Z}')$ such that $f|_{\mathbf{A}} = 1$ and $f|_{\mathbf{W}} = 0$. Take $\mathbf{V}$ to be $\mathbf{Z}'_f$.

Case 5. The general case.
Embed $\mathbf{Z}$ into a projective variety $\mathbf{Z}'$ as an open dense subset. By Case 3 we can find an open affine subset $\mathbf{V}' \subset \mathbf{Z}'$ satisfying $\mathbf{A} \subset \mathbf{V}'$. Note that $\mathbf{V}' \cap \mathbf{Z}$ is quasi-affine. The assertion follows now from the previous case.

∎

Corollary 3.1.7. *Let a finite group* Γ *act on a quasi-projective variety* $\mathbf{Z}$*. Then* $\mathbf{Z}$ *can be covered by* Γ*-invariant open affine subsets.*

[3] In fact, an accurate repetition of Case 1 in the language of schemes will give a proof for this case, however we prefer the more geometric approach below.

in view of Corollary 3.1.4, this gives:

Corollary 3.1.8. *An action of a finite group on a quasi-projective variety is factorizable.*

3.2. Quotients by free actions.

Lemma 3.2.1. *Let $\phi : \mathbf{X} \to \mathbf{Y}$ be a morphism of algebraic varieties. Assume that $\phi_{\bar{F}} : \mathbf{X}_{\bar{F}} \to \mathbf{Y}_{\bar{F}}$ is étale. Then so is ϕ.*

Proof. Without loss of generality we may assume that $\mathbf{X}$ and $\mathbf{Y}$ are affine.

Step 1. There exists a finite extension E/F such that $\phi_E : \mathbf{X}_E \to \mathbf{Y}_E$ is étale.
It is easy to see that for any E, ϕ_E is flat. For any E/F let

$$I_E := \ker(\mathcal{O}_{\mathbf{X}}(\mathbf{X}) \otimes_{\mathcal{O}_{\mathbf{Y}}(\mathbf{Y})} \mathcal{O}_{\mathbf{X}}(\mathbf{X}) \to \mathcal{O}_{\mathbf{X}}(\mathbf{X})).$$

The fact that ϕ_E is unramified is equivalent to the fact that $I_E = I_E^2$. The assertion follows now from the fact that I_F is finitely generated (as guaranteed by the Hilbert basis theorem).

Step 2. ϕ is étale.
Let E be as in the previous step. It is easy to see that the natural map $\mathbf{Y}_E \to \mathbf{Y}$ is finite (and hence integral). The assertion follows now from descent for étale morphisms (see [Sta25, Proposition 41.20.6]).

□

Lemma 3.2.2. *Let $\phi : \mathbf{Z}_1 \to \mathbf{Z}_2$ be an étale map s.t. $\phi(\bar{F}) : \mathbf{Z}_1(\bar{F}) \to \mathbf{Z}_2(\bar{F})$ is a bijection. Then ϕ is an isomorphism.*

Proof. Step 1. Let $\psi : \mathbf{Z}_1 \to \mathbf{Z}_2$ be a standard étale map (see [Sta25, Definition 00UB]) s.t. $\psi(\bar{F}) : \mathbf{Z}_1(\bar{F}) \to \mathbf{Z}_2(\bar{F})$ is a injection. Then ψ is an open embedding.
Follows immediately from the definition.

Step 2. ϕ is an isomorphism.
By the previous step we have an open cover $\mathbf{Z}_1 = \bigcup \mathbf{U}_i$ s.t. $\phi|_{\mathbf{U}_i}$ is an open embedding. Since $\phi(\bar{F})$ is a bijection, we obtain that

$$\mathbf{Z}_2 = \bigcup \phi(\mathbf{U}_i).$$

Now we can define ϕ^{-1} on each $\phi(\mathbf{U}_i)$, and the compatibility follows from the fact that $\phi(\bar{F})$ is a bijection.

□

Lemma 3.2.3. *Let a finite group Γ act factorizably on a variety $\mathbf{Z}$. Assume that the action of Γ on $\mathbf{Z}$ is free (i.e. the action of Γ on $\mathbf{Z}(\bar{F})$ is free). Then*

(i) The map $\mathbf{Z} \to \mathbf{Z}//\Gamma$ is étale.
(ii) The natural morphism $m : \mathbf{Z} \times \Gamma \to \mathbf{Z} \times_{\mathbf{Z}//\Gamma} \mathbf{Z}$ is an isomorphism.

Proof.

(i) Follows from [Mum08, §II.7] and Lemma 3.2.1.

(ii)

Step 1. The map m it is étale.
It is enough to show that $m|_{\mathbf{Z}\times\{1\}}$ is étale. This map is the diagonal map $\Delta : \mathbf{Z} \to \mathbf{Z} \times_{\mathbf{Z}//\Gamma} \mathbf{Z}$, which is étale by (i) (as the diagonal of an étale map is étale – [Sta25, Lemma 02GE]).

Step 2. m induces a bijection on the $\bar{F}$-points.
Follows from Lemma 3.0.2 since the action of Γ is free.

Step 3. m is an isomorphism.
Follows from the previous steps using Lemma 3.2.2.

□

Corollary 3.2.4 (Galois descent for free actions)**.** *In the setting of the previous lemma, let $Sch_{\mathbf{Z}//\Gamma}$ denote the category of schemes over $\mathbf{Z}//\Gamma$ and $Sch_{\mathbf{Z}}^{\Gamma}$ denote the category of schemes over $\mathbf{Z}$ equipped with an action of Γ which is compatible with the action of Γ on $\mathbf{Z}$. Consider the functor $\mathcal{F} : Sch_{\mathbf{Z}//\Gamma} \to Sch_{\mathbf{Z}}^{\Gamma}$ defined by $\mathcal{F}(\mathbf{X}) = \mathbf{X} \times_{\mathbf{Z}//\Gamma} \mathbf{Z}$, with Γ acting on the second coordinate. Let $\beta : \mathcal{F}(\mathbf{X}) \to \mathbf{X}$ be the projection on the first coordinate. Then*

(i) $\mathcal{F}$ is fully faithful.

(ii) Given $\mathbf{X} \in Sch_{\mathbf{Z}//\Gamma}$ and a sheaf $\mathcal{V}$ on it, the pullback $\mathcal{V}(\mathbf{X}) \to (\beta^*\mathcal{V})(\mathcal{F}(\mathbf{X}))$ with respect to β gives an isomorphism

$$\mathcal{V}(\mathbf{X}) \cong (\beta^*\mathcal{V})(\mathcal{F}(\mathbf{X}))^{\Gamma}.$$

Proof.

(i) Let $\mathbf{X}_1, \mathbf{X}_2 \in Sch_{\mathbf{Z}//\Gamma}$. We need to show that $\mathcal{F}$ induces a bijection

$$Mor_{Sch_{\mathbf{Z}}^{\Gamma}}(\mathcal{F}(\mathbf{X}_1), \mathcal{F}(\mathbf{X}_2)) \to Mor_{Sch_{\mathbf{Z}//\Gamma}}(\mathbf{X}_1, \mathbf{X}_2).$$

The previous lemma implies that:

(8) the maps $\mathcal{F}(\mathbf{X}_i) \to \mathbf{X}_i$ are étale (and surjective)

(9) the natural maps $\mathcal{F}(\mathbf{X}_i) \times \Gamma \to \mathcal{F}(\mathbf{X}_i) \times_{\mathbf{Z}//\Gamma} \mathcal{F}(\mathbf{X}_i)$ are isomorphisms.

The assertion follows now from faithfully flat descent for morphisms, see *e.g.* [Tsi14, Lecture 9, Theorem 1.1].

(ii) Follows from (8,9), using the fact that a coherent sheaf in the Zariski topology is also a sheaf in the étale topology, see *e.g.* [Sta25, Lemma 03DT].

□

4. Proof of Theorem B

We will use the following straightforward criterion for sharp integrability:

Lemma 4.0.1. *Let $\phi : \tilde{\mathbf{Z}} \to \mathbf{Z}$ be a modification. Assume that*

(a) $\mathbf{Z}^{sm}$ is big in $\mathbf{Z}$,

(b) $\mathbf{Z}^{sm}$ admits an invertible top form ω, and

(c) $\phi^*(\omega)$ can be extended to an invertible top form on $\tilde{\mathbf{Z}}^{sm}$.

Then ϕ is sharply integrable.

We need the following notation:

Notation 4.0.2. *Let $\mathbf{Z}$ be a smooth algebraic (quasi-projective) surface and $x \in Z := \mathbf{Z}(F)$. Define the following:*

- *Let $\iota_{\mathbf{Z},n} : \mathbf{Z}^n \to \mathbf{Z}^{(n)}$ denote the quotient map.*
- *$\Delta^n_{\mathbf{Z}} \subset \mathbf{Z}^n$ the diagonal copy.*
- *$\mathbf{Z}^{(n)}_{diag} := \iota_{\mathbf{Z},n}(\Delta^n_{\mathbf{Z}}) \subset \mathbf{Z}^{(n)}$. Note that it is closed since ι_Z is finite.*
- *$\mathbf{Z}^{[n]}_{diag} := \mathfrak{H}^{-1}_{\mathbf{Z},n}(\mathbf{Z}^{(n)}_{diag}) \subset \mathbf{Z}^{[n]}$.*
- *$\mathbf{Z}^{(n)}_x := \iota_{\mathbf{Z},n}(\{(x, \dots, x)\}) \subset \mathbf{Z}^{(n)}_{diag}$ and $\mathbf{Z}^{[n]}_x := \mathfrak{H}^{-1}_{\mathbf{Z},n}(\mathbf{Z}^{(n)}_x) \subset \mathbf{Z}^{[n]}_{diag}$.*

Proposition 4.0.3 ([BK05, 7.4.E.3])**.** *Let $\mathbf{Z}$ be a smooth algebraic (quasi-projective) surface. Then for any $x \in \mathbf{Z}(F)$ we have $\dim \mathbf{Z}^{[n]}_x = n - 1$.*

Together with Theorem 1.1.3 this proposition gives the following corollary.

Corollary 4.0.4. *Let $\mathbf{Z}$ be a smooth algebraic (quasi-projective) surface. Then*
$$\dim \mathbf{Z}^{[n]} - \dim \mathbf{Z}^{[n]}_{diag} = n - 1.$$

Notation 4.0.5. *Write $n = n_1 + n_2$. Let $\mathbf{Z}$ be a quasi-projective algebraic variety. Denote*
$$\mathbf{Z}^{n_1,n_2} := \{(z_1, \dots, z_n) \in \mathbf{Z}^n | \, \{z_1, \dots, z_{n_1}\} \cap \{z_{n_1+1}, \dots, z_n\} = \emptyset\}.$$
Denote also
$$\mathbf{Z}^{(n_1,n_2)} := \mathbf{Z}^{n_1,n_2} // (S_{n_1} \times S_{n_2})$$
and by
$$\iota_{\mathbf{Z},n_1,n_2} : \mathbf{Z}^{n_1,n_2} \to \mathbf{Z}^{(n_1,n_2)}$$
the quotient map.

Lemma 4.0.6 (See §4.1 below)**.** *Let $\mathbf{Z}$ be a quasi-projective algebraic variety. Write $n = n_1 + n_2$. Then there exist*

- *a variety $\mathbf{Z}^{[n_1,n_2]}$*
- *morphisms of varieties $\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}, \mathcal{C}^{\mathbf{Z}}_{(n_1,n_2)}, \mathcal{C}^{\mathbf{Z}}_{n_1,n_2}, \mathfrak{H}_{\mathbf{Z},n_1,n_2}$*
- *an open embedding $\mathbf{Z}^{[n_1,n_2]} \subset \mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]}$*

s.t.

- *We have the following commutative diagram:*

(10)
$$\begin{array}{ccccc}
\mathbf{Z}^{[n]} & \xleftarrow{\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}} & \mathbf{Z}^{[n_1,n_2]} & \subset & \mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]} \\
\downarrow{\scriptstyle \mathfrak{H}_{\mathbf{Z},n}} & & \downarrow{\scriptstyle \mathfrak{H}_{\mathbf{Z},n_1,n_2}} & \square & \downarrow{\scriptstyle \mathfrak{H}_{\mathbf{Z},n_1} \times \mathfrak{H}_{\mathbf{Z},n_2}} \\
\mathbf{Z}^{(n)} & \xleftarrow{\mathcal{C}^{\mathbf{Z}}_{(n_1,n_2)}} & \mathbf{Z}^{(n_1,n_2)} & \subset & \mathbf{Z}^{(n_1)} \times \mathbf{Z}^{(n_2)} \\
\uparrow{\scriptstyle \iota_{\mathbf{Z},n}} & & \uparrow{\scriptstyle \iota_{\mathbf{Z},n_1,n_2}} & \square & \uparrow{\scriptstyle \iota_{\mathbf{Z},n_1} \times \iota_{\mathbf{Z},n_2}} \\
\mathbf{Z}^{n} & \xleftarrow{\mathcal{C}^{\mathbf{Z}}_{n_1,n_2}} & \mathbf{Z}^{n_1,n_2} & \subset & \mathbf{Z}^{n_1} \times \mathbf{Z}^{n_2}
\end{array}$$
$$\mathbf{Z}^{n} \xrightarrow{\cong} \mathbf{Z}^{n_1} \times \mathbf{Z}^{n_2}$$

- *The embeddings in the diagram are open.*
- $\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}$ *is étale,*
- *the top left square in the diagram is Cartesian on the level of* $\bar{F}$ *points.*
- *The bottom curved arrow is the standard identification* $\mathbf{Z}^n \cong \mathbf{Z}^{n_1} \times \mathbf{Z}^{n_2}$.

Notation 4.0.7. *Note that* $\mathbf{Z}^{[n_1,n_2]}, \mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}, \mathcal{C}^{\mathbf{Z}}_{(n_1,n_2)}, \mathcal{C}^{\mathbf{Z}}_{n_1,n_2}, \mathfrak{H}_{\mathbf{Z},n_1,n_2}$ *are defined uniquely by the previous lemma, so we will use these notations in the rest of the section.*

Notation 4.0.8. *Let* $\mathbf{Z}$ *be a quasi-projective algebraic variety. Denote*

(i) $\mathbf{Z}^n_0 := \{(z_1, \ldots, z_n) \in \mathbf{Z}^n \mid \forall i, j, z_i \neq z_j\}$.

(ii) $\mathbf{Z}^{(n)}_0 := \iota_n(\mathbf{Z}^n_0) \subset \mathbf{Z}^{(n)}$. *By Corollary 3.1.5, it is an open subset.*

(iii) $\mathbf{Z}^{(n_1,n_2)}_0 := \iota_{n_1} \times \iota_{n_2}(\mathbf{Z}^n_0) \subset \mathbf{Z}^{(n_1,n_2)}$.

From Lemma 4.0.6 we obtain the following corollary.

Corollary 4.0.9. *Let* $\mathbf{Z}$ *be a quasi-projective algebraic variety. Then we have*

(i) $\mathcal{C}^{\mathbf{Z}}_{n_1,n_2}(\mathbf{Z}^{(n_1,n_2)}_0) = \mathbf{Z}^{(n)}_0$.

(ii) $\mathcal{C}^{\mathbf{Z}}_{n_1,n_2}|_{\mathbf{Z}^{(n_1,n_2)}_0}$ *is an étale map.*

(iii) $\mathbf{Z}^{(n)}_0 \subset \mathbf{Z}^{(n)}$ *and* $\mathbf{Z}^{(n_1,n_2)}_0 \subset \mathbf{Z}^{(n_1,n_2)}$ *are big subsets.*

(iv) The image $\mathcal{C}^{\mathbf{Z}}_{n_1,n_2}(\mathbf{Z}^{(n_1,n_2)})$ *is open.*

(v) $\bigcup_{n_1=1}^{n-1} \mathcal{C}^{\mathbf{Z}}_{n_1,n_2}(\mathbf{Z}^{(n)}) = \mathbf{Z}^{(n)} \smallsetminus \mathbf{Z}^{(n)}_{diag}$.

The following follows from the Zariski main theorem:

Lemma 4.0.10. *Let* $\gamma : \mathbf{Z}_1 \to \mathbf{Z}_2$ *be a morphism of algebraic varieties. Assume that:*

- $\mathbf{Z}_i$ *are irreducible.*
- $\mathbf{Z}_2$ *normal.*
- γ *induces a bijection:* $\mathbf{Z}_1(\bar{F}) \to \mathbf{Z}_2(\bar{F})$.

Then γ *is an isomorphism.*

Proof. Notice that γ is dominant and the fibers over geometric points are singletons. Hence γ is birational. Also γ is quasi-finite. By Zariski main theorem we can decompose: $\gamma = \pi \circ j$ with $j : \mathbf{Z}_1 \to \mathbf{Z}_3, \pi : \mathbf{Z}_3 \to \mathbf{Z}_2$ where π is finite and j is an open immersion. As $\mathbf{Z}_1$ is normal, and finite birational morphism onto a normal variety is an isomorphism, it follows that π is an isomorphism. As the image of j must contain all geometric points, it is easy to see that j is also an isomorphism and we are done. □

Lemma 4.0.11. *Let* $\mathbf{Z} = \mathbb{A}^2$. *Then there is a commutative diagram:*

(11)
$$\begin{array}{ccc} Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2 & \xrightarrow{bl} & \mathbf{Z}^2 \\ \downarrow{\scriptstyle q_{S_2}} & & \downarrow{\scriptstyle \iota_{\mathbf{Z},2}} \\ \mathbf{Z}^{[2]} & \xrightarrow{\mathfrak{H}_{\mathbf{Z},2}} & \mathbf{Z}^{(2)} \end{array}$$

where the top row is the blowing-up of $\mathbf{Z}^2$ *along the diagonal* $\Delta_{\mathbf{Z}}$*, and the left vertical arrow is the quotient map by the action of* S_2 *given by the flip of the 2 copies of* $\mathbf{Z}$*.*

Proof. First let us construct the map q_{S_2}. By the definition of the Hilbert scheme $\mathbf{Z}^{[2]}$ this means to construct a scheme $\mathbf{Y} \subset (Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2) \times \mathbf{Z}^2$ which is finite flat of rank 2 over $Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2$. Realize $Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2$ as

$$\{(l, x, y) | l \text{ is a line in } \mathbf{Z}; x, y \in l\}$$

We get

$$(Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2) \times \mathbf{Z} = \{(l, x, y, z) | l \text{ is a line in } \mathbf{Z}; x, y \in l; z \in \mathbf{Z}\}$$

Let $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ be the sheaves of ideals in $(Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2) \times \mathbf{Z}$ given by the conditions:

1. $x = z$
2. $y = z$
3. $z \in l$

respectively. Define $\mathcal{I} := \langle \mathcal{I}_1\mathcal{I}_2, \mathcal{I}_3 \rangle$, and let $\mathbf{Y}$ be its 0-locus. It is easy to see that $\mathbf{Y}$ is finite flat of rank 2 over $Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2$ and thus defines a map $q_{S_2} : Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2 \to \mathbf{Z}^{[2]}$. By Corollary 3.1.8 there exists a categorical quotient $Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2//S_2$. So q_{S_2} factors through a map $\gamma : Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2//S_2 \to \mathbf{Z}^{[2]}$. It is easy to see that this map is a bijection on the level of $\bar{F}$ points. Also, by Theorem 1.1.3 $\mathbf{Z}^{[2]}$ is smooth and irreducible. Hence Lemma 4.0.10 implies that γ is an isomorphism.

So we constructed the diagram (11) and proved that q_{S_2} is the quotient map by the action of S_2. It is left to show that this diagram is commutative. It is enough to verify it on the level of $\bar{F}$ points. This follows from the definitions. □

The following lemma is obvious.

Lemma 4.0.12. *Let* $\gamma : \mathbf{X} \to \mathbf{Y}$ *be a modification of algebraic varieties. Let* $\mathbf{U} \subset \mathbf{Y}$ *be an open set. Assume that:*

- $\gamma^{-1}(\mathbf{U})$ *is big in* $\mathbf{X}$
- $\gamma|_{\gamma^{-1}(\mathbf{U})} \to \mathbf{U}$ *is a (sharply) integrable modification.*

Then $\gamma : \mathbf{X} \to \mathbf{Y}$ *is a (sharply) integrable modification.*

Lemma 4.0.13. *Assume that we have a commutative diagram*

$$\begin{array}{ccc} \mathbf{Z}_{11} & \xleftarrow{e_1} & \mathbf{Z}_{12} \\ {\scriptstyle m_1}\downarrow & & \downarrow{\scriptstyle m_2} \\ \mathbf{Z}_{21} & \xleftarrow{e_2} & \mathbf{Z}_{22} \end{array}$$

Assume that e_1 *is onto and étale,* e_2 *is of relative dimension zero,* m_i *are resolutions of singularities,* m_2 *is an integrable modification, and there exists an open smooth set* $\mathbf{U} \subset \mathbf{Z}_{21}$ *such that* $e_2^{-1}(\mathbf{U}) \subset \mathbf{Z}_{22}$ *is a big subset. Then* m_1 *is an integrable modification.*

Proof. Let $\mathbf{V} \subset \mathbf{Z}_{21}$ be an open set, and ω be a regular top form on the smooth locus $\mathbf{V}^{sm}$ of $\mathbf{V}$. We have to show that $m_1^*(\omega)$ can be extended to a top form on $m_1^{-1}(\mathbf{V})$. Since e_1 is onto étale, it is enough to show that $e_1^* m_1^*(\omega)$ can be extended to a top form on $e_1^{-1} m_1^{-1}(\mathbf{V})$. Equivalently, it is enough to show that $m_2^* e_2^*(\omega)$ can be extended to a top form on $m_2^{-1} e_2^{-1}(\mathbf{V})$. Note that $e_2^*(\omega)$ is regular on $e_2^{-1}(\mathbf{V}^{sm} \cap \mathbf{U}) = e_2^{-1}(\mathbf{V}) \cap e_2^{-1}(\mathbf{U})$. Since $e_2^{-1}(\mathbf{U})$ is big in $\mathbf{Z}_{22}$ this implies that $e_2^*(\omega)$ can be extended to a regular form on the smooth locus of $e_2^{-1}(\mathbf{V})$. Now, the fact that m_2 is integrable implies the assertion. ☐

Proof of Theorem B. Let $\mathbf{Z} := \mathbb{A}^2$. By Corollary 1.3.4 the Hilbert-Chow map is a resolution of singularities. So we need to show that it is integrable. We will do it by analyzing the following cases.

Case 1. $n = 2$.

Let $\mathbf{U} \subset \mathbf{Z}^2$ be the complement to the diagonal. By Corollary 3.1.5 $\iota_{\mathbf{Z},2}(\mathbf{U})$ is open in $\mathbf{Z}^{(2)}$, and $\iota_{\mathbf{Z},2}(\mathbf{U}) \cong \mathbf{U}//S_2$. By Lemma 3.2.3, $\mathbf{U}//S_2$ is smooth. We obtain that $\mathbf{V} := \iota_{\mathbf{Z},2}(\mathbf{U})$ is an open subset of the smooth locus of $\mathbf{Z}^{(2)}$. In fact, it is equal to this smooth locus. Also, $\iota_{\mathbf{Z},2}^{-1}(\mathbf{V}) \subset \mathbf{Z}^2$ is big. Let $\omega_{\mathbf{Z}^2}$ be the standard top form on $\mathbf{Z}^2 = \mathbb{A}^4$. It is easy to see that it is S_2-invariant. Thus, by Corollary 3.2.4, there exists a top form $\omega_{\mathbf{V}}$ s.t. $\iota_{\mathbf{Z},2}^*(\omega_{\mathbf{V}}) = \omega_{\mathbf{Z}^2}|_{\mathbf{U}}$. It is easy to see that $\omega_{\mathbf{V}}$ is an invertible form. Consider $\omega_{\mathbf{V}}$ as a rational top form on $\mathbf{Z}^{(2)}$.

By Lemma 4.0.1 it is enough to show that $\mathfrak{H}_{\mathbf{Z},2}^*(\omega_{\mathbf{V}})$ is an invertible top form on $\mathbf{Z}^{[2]}$.

By Lemma 4.0.11 we have the following commutative diagram:

$$\begin{array}{ccc} Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2 & \xrightarrow{bl} & \mathbf{Z}^2 \\ \downarrow{\scriptstyle q_{S_2}} & & \downarrow{\scriptstyle \iota_{\mathbf{Z},2}} \\ \mathbf{Z}^{[2]} & \xrightarrow{\mathfrak{H}_{\mathbf{Z},2}} & \mathbf{Z}^{(2)} \end{array}$$

where the top row is the blowing-up and the left vertical arrow is the quotient map by the action of S_2.

The assertion follows now from the following 2 statements:

(a) For an invertible form ω on $\mathbf{Z}^2$, the zero locus of the form $bl^*(\omega)$ is the divisor $bl^{-1}(\Delta_{\mathbf{Z}})$ with multiplicity 1.

(b) If ω is a rational form on $\mathbf{Z}^{[2]} \cong Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2//S_2$ s.t. $q_{S_2}^*(\omega)$ is a regular form and its zero locus is the divisor $bl^{-1}(\Delta_{\mathbf{Z}})$ with multiplicity 1 then ω is regular.

Proof of (a): This is a standard property of a blowing up of a smooth variety along smooth subvariety of co-dimension 2.

Proof of (b): As in the proof of Lemma 4.0.11 we can realize $Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2$ as $\{(l, x, y) | l \text{ is a line in } \mathbf{Z}; x, y \in l\}$. This gives a map $Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2 \to \mathcal{L}$, where $\mathcal{L}$ is the collection of lines in $\mathbf{Z} = \mathbb{A}^2$. This map is S_2-invariant, so we get a commutative diagram:

$$\begin{array}{ccc} Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2 & & \\ \downarrow{\scriptstyle q_{S_2}} & \searrow & \\ (Bl_{\Delta_{\mathbf{Z}}}\mathbf{Z}^2)//S_2 & \longrightarrow & \mathcal{L} \end{array} \tag{12}$$

The statement is Zariski local on $\mathcal{L}$. Let $\tilde{\mathcal{L}} = \bigsqcup \mathcal{L}_i \to \mathcal{L}$ be a Zariski cover that trivializes the tautological bundle. When we pull the diagram (12) to $\tilde{L}$ we obtain a diagram isomorphic to:

$$\begin{array}{ccc} \tilde{\mathcal{L}} \times \mathbb{A}^2 & & \\ \downarrow & \searrow & \\ \tilde{\mathcal{L}} \times \mathbb{A}^2//S_2 & \longrightarrow & \tilde{\mathcal{L}} \end{array} \tag{13}$$

where S_2 acts on $\mathbb{A}^2$ by flipping the coordinates.
Let $q'_{S_2} : \mathbb{A}^2 \to \mathbb{A}^2//S_2$ be the quotient map.
It is enough to show that if η is a rational form on $\mathbb{A}^2//S_2$ s.t. $(q'_{S_2})^*(\eta)$ is regular and its zero locus is he divisor $\Delta_{\mathbb{A}^1} \subset \mathbb{A}^2$ with multiplicity 1 then η is regular. This is a straightforward computation.

Case 2. The general case. We prove the statement by induction on n. Case $n = 1$ is obvious. Case $n = 2$ is the previous case. Assume $n > 2$. By Proposition 4.0.3 we have $\dim \mathbf{Z}^{(n)} - \dim \mathbf{Z}^{(n)}_{diag} > 1$. Thus by Lemma 4.0.12 it is enough to prove that

$$\mathfrak{H}_{\mathbf{Z},n}|_{\mathbf{Z}^{(n)}} : \mathbf{Z}^{(n)} \smallsetminus \mathbf{Z}^{(n)}_{diag} \to \mathbf{Z}^{[n]} \smallsetminus \mathbf{Z}^{[n]}_{diag}$$

is an integrable modification. Write $n = n_1 + n_2$ with $n_1, n_2 > 0$. Denote $\mathbf{U} := \mathcal{C}^{\mathbf{Z}}_{n_1,n_2}(\mathbf{Z}^{(n_1,n_2)}) \subset \mathbf{Z}^{(n)}$. By Corollary 4.0.9(iv), it is an open subscheme. Denote $\mathbf{V} := \mathfrak{H}^{-1}_{\mathbf{Z},n}(\mathbf{U})$. By Corollary 4.0.9(v), it is enough to show that

$$\mathfrak{H}_{\mathbf{Z},n}|_{\mathbf{V}} : \mathbf{V} \to \mathbf{U}$$

is an integrable modification, for any decomposition $n = n_1 + n_2$. Let $\mathfrak{H}_{\mathbf{Z},n_1,n_2} : \mathbf{Z}^{[n_1,n_2]} \to \mathbf{Z}^{(n_1,n_2)}$ be as in Lemma 4.0.6. Lemma 4.0.6 and the induction hypothesis imply that $\mathfrak{H}_{\mathbf{Z},n_1,n_2}$ is an integrable modification. By Lemma 4.0.6, the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{V} & \xleftarrow{\overset{\circ}{\mathcal{C}}_{[n_1,n_2]}} & \mathbf{Z}^{[n_1,n_2]} \\ {\scriptstyle \overset{\circ}{\mathfrak{H}}_{\mathbf{Z},n}}\downarrow & & \downarrow{\scriptstyle \mathfrak{H}_{\mathbf{Z},n_1,n_2}} \\ \mathbf{U} & \xleftarrow{\overset{\circ}{\mathcal{C}}_{(n_1,n_2)}} & \mathbf{Z}^{(n_1,n_2)} \end{array},$$

where the maps $\overset{\circ}{\mathcal{C}}_{[n_1,n_2]}$, $\overset{\circ}{\mathcal{C}}_{(n_1,n_2)}$, and $\overset{o}{\mathfrak{H}}_{\mathbf{Z},n}$ are obtained by restriction from the maps $\mathcal{C}_{[n_1,n_2]}$ and $\mathfrak{H}_{\mathbf{Z},n}$. Moreover, this diagram is a Cartesian square on the level of $\bar{F}$-points. This implies that the map $\overset{\circ}{\mathcal{C}}_{[n_1,n_2]}$ is onto. By Lemma 4.0.6, it is also étale. By Corollary 4.0.9(iii) and Lemma 4.0.13, $\mathfrak{H}_{\mathbf{Z},n}|_{\mathbf{V}} : \mathbf{V} \to \mathbf{U}$ is an integrable modification, as required.

□

4.1. **Proof of Lemma 4.0.6.** Note that by Corollary 3.1.5, we have the Cartesian square

$$
\begin{array}{ccc}
\mathbf{Z}^{(n_1,n_2)} & \subset & \mathbf{Z}^{(n_1)} \times \mathbf{Z}^{(n_2)} \\
{\scriptstyle \iota_{\mathbf{Z},n_1,n_2}}\uparrow & \square & \uparrow{\scriptstyle \iota_{\mathbf{Z},n_1}\times\iota_{\mathbf{Z},n_2}} \\
\mathbf{Z}^{n_1,n_2} & \subset & \mathbf{Z}^{n_1} \times \mathbf{Z}^{n_2}
\end{array} \tag{14}
$$

Denote $\mathbf{Z}^{[n_1,n_2]} := (\mathfrak{H}_{\mathbf{Z},n_1} \times \mathfrak{H}_{\mathbf{Z},n_2})^{-1}(\mathbf{Z}^{(n_1,n_2)})$. This gives us the Cartesian square

$$
\begin{array}{ccc}
\mathbf{Z}^{[n_1,n_2]} & \subset & \mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]} \\
{\scriptstyle \mathfrak{H}_{\mathbf{Z},n_1,n_2}}\downarrow & \square & \downarrow{\scriptstyle \mathfrak{H}_{\mathbf{Z},n_1}\times\mathfrak{H}_{\mathbf{Z},n_2}} \\
\mathbf{Z}^{(n_1,n_2)} & \subset & \mathbf{Z}^{(n_1)} \times \mathbf{Z}^{(n_2)}
\end{array} \tag{15}
$$

Definition 4.1.1. *For an algebraic variety* $\mathbf{Z}$ *define the subfunctor*

$$Hilb_{n_1,n_2}(\mathbf{Z}) \subset Hilb_{n_1}(\mathbf{Z}) \times Hilb_{n_2}(\mathbf{Z}) : Sch_F^{op} \to sets$$

by

$$Hilb_{n_1,n_2}(\mathbf{Z})(\mathbf{S}) := \{(\mathbf{Y}_1, \mathbf{Y}_2) \in Hilb_{n_1}(\mathbf{Z})(\mathbf{S}) \times Hilb_{n_2}(\mathbf{Z})(\mathbf{S}) \mid \mathbf{Y}_1 \cap \mathbf{Y}_2 = \emptyset\}$$

Lemma 4.1.2. *The subfunctor* $Hilb_{n_1,n_2}$ *is represented by the open subscheme* $\mathbf{Z}^{[n_1,n_2]}$.

For the proof we will need the following straightforward lemma:

Lemma 4.1.3. *Consider the following commutative diagram in arbitrary category.*

$$
\begin{array}{ccccc}
Z_{11} & \xrightarrow{\gamma_{11}} & Z_{12} & \xrightarrow{\gamma_{12}} & Z_{13} \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & \square & \downarrow{\scriptstyle \delta_3} \\
Z_{21} & \xrightarrow{\gamma_{21}} & Z_{22} & \xrightarrow{\gamma_{22}} & Z_{23}
\end{array}
$$

Assume also that we have:

$$
\begin{array}{ccc}
Z_{11} & \xrightarrow{\gamma_{12}\circ\gamma_{11}} & Z_{13} \\
\downarrow{\scriptstyle \delta_1} & \square & \downarrow{\scriptstyle \delta_3} \\
Z_{21} & \xrightarrow{\gamma_{22}\circ\gamma_{21}} & Z_{23}
\end{array}
$$

Then we have:

$$\begin{array}{ccc} Z_{11} & \xrightarrow{\gamma_{11}} & Z_{12} \\ \downarrow{\scriptstyle \delta_1} & \square & \downarrow{\scriptstyle \delta_2} \\ Z_{21} & \xrightarrow{\gamma_{21}} & Z_{22} \end{array}$$

Proof of Lemma 4.1.2.

Step 1. $Hilb_{n_1,n_2}$ is represented by an open subscheme of $\mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]}$.

Step a. Construction of $\mathbf{Z}_0^{[n_1,n_2]}$.
Let $\tilde{\mathbf{Z}}^{[n_i]} \subset \mathbf{Z} \times \mathbf{Z}^{[n_i]}$ be the tautological scheme over $\mathbf{Z}^{[n_i]}$, i.e. the subscheme of $\mathbf{Z} \times \mathbf{Z}^{[n_i]}$ that corresponds to the identity map under the isomorphism

$$Mor(\mathbf{Z}^{[n_i]}, \mathbf{Z}^{[n_i]}) \cong Hilb_{[n_i]}(\mathbf{Z})(\mathbf{Z}^{[n_i]}).$$

Let $\mathcal{W} := \tilde{\mathbf{Z}}^{[n_1]} \times_{\mathbf{Z}} \tilde{\mathbf{Z}}^{[n_2]}$. We have a natural embedding $\mathcal{W} \subset \mathbf{Z} \times \mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]}$. Let $pr : \mathcal{W} \to \mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]}$ be the projection. Note that it is finite. Let $\mathbf{Z}_0^{[n_1,n_2]} := \mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]} \smallsetminus pr(\mathcal{W})$, and consider it as an open subscheme of $\mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]}$.

Step b. Proof that $\mathbf{Z}_0^{[n_1,n_2]}$ represents the functor $Hilb_{n_1,n_2}(\mathbf{Z})$.
Note that for any $S \in Sch_F$ we have

$$Mor(\mathbf{S}, \mathbf{Z}_0^{[n_1,n_2]}) = \left\{\gamma \in Mor(\mathbf{S}, \mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]}) | \gamma^*(\mathcal{W}) = \emptyset\right\}, \tag{16}$$

where $\gamma^*(\mathcal{W})$ is the object that makes the following square Cartesian:

$$\begin{array}{ccc} \gamma^*(\mathcal{W}) & \longrightarrow & \mathcal{W} \\ \downarrow & \square & \downarrow{\scriptstyle pr} \\ \mathbf{S} & \xrightarrow{\gamma} & \mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]}. \end{array}$$

Write $\gamma = (\gamma_1, \gamma_2)$. The maps γ_i corresponds to a $\mathbf{Y}_i \in Hilb_{n_i}(\mathbf{Z})(\mathbf{S})$. We will show that $\gamma^*(\mathcal{W}) = \mathbf{Y}_1 \cap \mathbf{Y}_2$.
We have Cartesian squares:

$$\begin{array}{ccc} \mathbf{Y}_i & \longrightarrow & \tilde{\mathbf{Z}}^{[n_i]} \\ \downarrow & \square & \downarrow{\scriptstyle pr} \\ \mathbf{S} & \xrightarrow{\gamma} & \mathbf{Z}^{[n_i]}. \end{array} \tag{17}$$

Consider the following diagram:

$$\begin{array}{ccccc}
\mathbf{Y}_1 \cap \mathbf{Y}_2 & \longrightarrow & \mathcal{W} & \longrightarrow & \mathbf{Z} \\
\downarrow & 1 & \downarrow{\scriptstyle pr} & 2 & \downarrow{\scriptstyle diag} \\
\mathbf{Y}_1 \times_{\mathbf{S}} \mathbf{Y}_2 & \longrightarrow & \tilde{\mathbf{Z}}^{[n_1]} \times \tilde{\mathbf{Z}}^{[n_2]} & \longrightarrow & \mathbf{Z} \times \mathbf{Z} \\
\downarrow & 3 & \downarrow & & \\
\mathbf{S} & \xrightarrow{\gamma} & \mathbf{Z}^{[n_1]} \times \mathbf{Z}^{[n_2]} & &
\end{array}$$

where *diag* is the diagonal map. The square 3 is Cartesian because of (17). The square 2 is Cartesian by the definition of $\mathcal{W}$. It is obvious that the square

$$\begin{array}{ccc}
\mathbf{Y}_1 \cap \mathbf{Y}_2 & \longrightarrow & \mathbf{Z} \\
\downarrow{\scriptstyle pr} & 1 \circ 2 & \downarrow{\scriptstyle diag} \\
\mathbf{Y}_1 \times_{\mathbf{S}} \mathbf{Y}_2 & \longrightarrow & \mathbf{Z} \times \mathbf{Z}
\end{array}$$

is Cartesian. Therefore, by Lemma 4.1.3 the square 1 is Cartesian and thus $\gamma^*(\mathcal{W}) = \mathbf{Y}_1 \cap \mathbf{Y}_2$ Together with (16) this completes the step.

Step 2. $Mor(\operatorname{Spec} \bar{F}, \mathbf{Z}^{[n_1,n_2]}) = Hilb_{n_1,n_2}(\mathbf{Z})(\operatorname{Spec} \bar{F})$.
Follows directly from the definitions of $\mathbf{Z}^{[n_1,n_2]}$ and $Hilb_{n_1,n_2}(\mathbf{Z})$, the characterization of the Hilbert-Chow map (Theorem 1.3.3) and Lemma 3.0.2.

Step 3. End of the proof.
Follows from the 2 previous steps and the fact that an open subset in a variety is determined by its $\bar{F}$ points. The later follows from the Nullstellensatz.

□

Notation 4.1.4. *Define a natural transformation*

$$\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]} : Hilb_{n_1,n_2}(\mathbf{Z}) \to Hilb_n(\mathbf{Z})$$

by

$$\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}(\mathbf{Y}_1, \mathbf{Y}_2) := \mathbf{Y}_1 \sqcup \mathbf{Y}_2$$

By Lemma 4.1.2 this morphism defines a morphism $\mathbf{Z}^{[n_1,n_2]} \to \mathbf{Z}^{[n]}$, *that we will also denote by* $\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}$.

Notation 4.1.5. *Let* $\overline{\mathcal{C}}^{\mathbf{Z}}_{(n_1,n_2)} : \mathbf{Z}^n//(S_{n_1} \times S_{n_2}) \to \mathbf{Z}^n//S_n = \mathbf{Z}^{(n)}$ *denote the natural map. By Corollary 3.1.5,* $\mathbf{Z}^{(n_1,n_2)}$ *can be considered as an open subset of* $\mathbf{Z}^n//(S_{n_1} \times S_{n_2})$. *Denote the restriction of* $\overline{\mathcal{C}}^{\mathbf{Z}}_{(n_1,n_2)}$ *to* $\mathbf{Z}^{(n_1,n_2)}$ *by* $\mathcal{C}^{\mathbf{Z}}_{(n_1,n_2)}$. *Finally, we let* $\mathcal{C}^{\mathbf{Z}}_{n_1,n_2} : \mathbf{Z}^{n_1,n_2} \to \mathbf{Z}^n$ *be the map given by concatenation of tuples.*

Now we defined all the arrows in the diagram (10). It remains to show that:

(a) The diagram is commutative.
(b) The top left square is Cartesian on the level of $\bar{F}$ points.
(c) The map $\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}$ is étale.

It is enough to prove (a) on the level of $\bar{F}$ points. Thus (a) and (b) are straightforward computations in view of Corollary 3.1.5, the definition of Hilbert scheme (Definition 1.1.1), and the characterization of the Hilbert-Chow map (Theorem 1.3.3).

It remains to show (c). By Theorem 1.1.3, the variety $\mathbf{Z}^{[n]}$ is smooth. By diagrams (14,15) the variety $\mathbf{Z}^{[n_1,n_2]}$ is an open subset of $\mathbf{Z}^{[n]}$, and thus is also smooth. It is enough to show that the map $\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}$ is étale at any closed point of $\mathbf{Z}^{[n_1,n_2]}$. Equivalently, we have to show that for any finite field extension E over F, and any $y \in \mathbf{Z}^{[n_1,n_2]}(E)$, the differential $d_y\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}$ is an isomorphism. Without loss of generality we assume that $E = F$ and $\mathbf{Z}$ is affine.

To $x \in \mathbf{Z}^{[n]}(F)$ we can assign an ideal $I \lhd \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})$. This gives us an identification

$$T_x\mathbf{Z}^{[n]} = \{\widetilde{I} \lhd \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})[t]/t^2 \mid (\mathcal{O}_{\mathbf{Z}}(\mathbf{Z})[t]/t^2)/\widetilde{I} \simeq (F[t]/t^2)^n \text{ and } \widetilde{I}/t = I\}$$

Here, the isomorphism is an isomorphism of $F[t]/t^2$-modules. Define

$$\gamma_{\mathbf{Z},n,x} : \mathrm{Hom}(I, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/I) \to T_x\mathbf{Z}^{[n]}$$

by $\gamma_{\mathbf{Z},n,x}(\varepsilon) = \{a + tb \mid a \in I, b \in \varepsilon(a)\}$. It is easy to see that $\gamma_{\mathbf{Z},n,x}$ is an isomorphism (cf. [BK05, Lemma 7.2.5]). Let

$$y = (x_1, x_2) \in \mathbf{Z}^{[n_1,n_2]}(F) \subset \mathbf{Z}^{[n_1]}(F) \times \mathbf{Z}^{[n_2]}(F).$$

We have to show that $d_y\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}$ is an isomorphism. Let $I_1, I_2 \lhd \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})$ be the ideals corresponding to the points x_1, x_2. The Chinese remainder theorem gives an identification

$$\delta_0 : \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/I_1 \oplus \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/I_2 \cong \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/(I_1 \cap I_2)$$

This gives an identification

$$\delta_1 : \mathrm{Hom}(I_1 \cap I_2, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/I_1 \oplus \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/I_2) \cong \mathrm{Hom}(I_1 \cap I_2, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/(I_1 \cap I_2))$$

Define a morphism

$$\delta : \mathrm{Hom}(I_1, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/I_1) \times \mathrm{Hom}(I_2, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/I_2) \to \mathrm{Hom}(I_1 \cap I_2, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/(I_1 \cap I_2))$$

by

$$\delta(\varepsilon_1, \varepsilon_2) := \delta_1(\varepsilon_1|_{I_1 \cap I_2}, \varepsilon_2|_{I_1 \cap I_2})$$

It is easy to see that the following diagram is commutative.

$$\begin{array}{ccccc}
\mathbf{T}_{x_1}\mathbf{Z}^{[n_1]} \times \mathbf{T}_{x_2}\mathbf{Z}^{[n_2]} & \cong & \mathbf{T}_y\mathbf{Z}^{[n_1,n_2]} & \xrightarrow{d_y\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}} & \mathbf{T}_{\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}(y)}\mathbf{Z}^{[n]} \\
\uparrow{\scriptstyle \gamma_{\mathbf{Z},n_1,x_1} \times \gamma_{\mathbf{Z},n_2,x_2}} \wr & & & & \uparrow{\scriptstyle \gamma_{\mathbf{Z},n,\mathcal{C}^{\mathbf{Z}}_{[n_1,n_2]}(y)}} \wr \\
\mathrm{Hom}(I_1, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/I_1) \times \mathrm{Hom}(I_2, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/I_2) & & \xrightarrow{\quad\delta\quad} & & \mathrm{Hom}(I_1 \cap I_2, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/(I_1 \cap I_2))
\end{array}$$

Thus it is enough to show that δ is an isomorphism. Let $\mathcal{I}_1, \mathcal{I}_2 \lhd \mathcal{O}_{\mathbf{Z}}$ be sheaves of ideals corresponding to the ideals I_1, I_2. δ defines a morphism of sheaves

$$\widetilde{\delta} : \mathcal{H}om(\mathcal{I}_1, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/\mathcal{I}_1) \times \mathcal{H}om(\mathcal{I}_2, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/\mathcal{I}_2) \to \mathcal{H}om(\mathcal{I}_1 \cap \mathcal{I}_2, \mathcal{O}_{\mathbf{Z}}(\mathbf{Z})/(\mathcal{I}_1 \cap \mathcal{I}_2)),$$

where $\mathcal{H}om$ denotes internal Hom of sheaves. It is enough to show that $\widetilde{\delta}$ is an isomorphism. Let U_i be the complement to the zero locus of $\mathcal{I}$. It is enough to show that $\delta|_{U_i}$ is an isomorphism for $i = 1, 2$. This is obvious.

5. Proof of Theorem A

Theorem A follows from Theorem B and the following lemma:

Lemma 5.0.1. *$((\mathbb{A}^2)^{(n)})^{sm}$ admits an invertible top form.*

Proof. Let $\mathbf{Z} = \mathbb{A}^2$. Let $(\mathbf{Z}^n)^0 \subset \mathbf{Z}^n$ be the open set of tuples of pairwise different points in $\mathbf{Z}$. By Corollary 3.1.5 we have the following Cartesian square:

$$\begin{array}{ccc} (\mathbf{Z}^n)^0 & \subset & \mathbf{Z}^n \\ \downarrow \iota^0_{\mathbf{Z},n} & \square & \downarrow \iota_{\mathbf{Z},n} \\ (\mathbf{Z}^{(n)})^0 & \subset & \mathbf{Z}^{(n)} \end{array}$$

with the horizontal inclusions being open. Let $\omega_{\mathbf{Z}^n}$ be the standard form on $\mathbf{Z}^n$ and let $\omega_{(\mathbf{Z}^n)^0}$ be its restriction to $(\mathbf{Z}^n)^0$. By Lemma 3.2.3 the map $\iota^0_{\mathbf{Z},n}$ is étale. So, $\Omega^{top}((\mathbf{Z}^n)^0) \cong (\iota^0_{\mathbf{Z},n})^*(\Omega^{top}((\mathbf{Z}^{(n)})^0))$

Note that $\omega_{(\mathbf{Z}^n)^0}$ is S_n invariant. So by Corollary 3.2.4 it gives a top form $\omega_{(\mathbf{Z}^{(n)})^0}$ on $(\mathbf{Z}^{(n)})^0$ s.t. $(\iota^0_{\mathbf{Z},n})^*(\omega_{(\mathbf{Z}^{(n)})^0}) = \omega_{(\mathbf{Z}^n)^0}$.

Since $\iota^0_{\mathbf{Z},n}$ is étale, the fact that $\omega_{(\mathbf{Z}^n)^0}$ is invertible implies that $\omega_{(\mathbf{Z}^{(n)})^0}$ is invertible. It is easy to see that $(\mathbf{Z}^{(n)})^0$ is big in $\mathbf{Z}^{(n)}$. Thus $\omega_{(\mathbf{Z}^{(n)})^0}$ can be extended to an invertible top form on $(\mathbf{Z}^{(n)})^{sm}$ as required. □

Index

References

[AA16] Avraham Aizenbud and Nir Avni. Representation growth and rational singularities of the moduli space of local systems. *Invent. Math.*, 204(1):245–316, 2016.

[AGKS] Avraham Aizenbud, Dmitry Gourevitch, David Kazhdan, and Eitan Sayag. On Harish-Chandra's integrability theorem in positive characteristic. Preprint available at https://www.wisdom.weizmann.ac.il/∼dimagur/.

[BK05] Michel Brion and Shrawan Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2005.

[Bou87] Jean-Fran¸cois Boutot. Singularités rationnelles et quotients par les groupes réductifs. *Invent. Math.*, 88(1):65–68, 1987.

[Gro62] Alexander Grothendieck. *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*. Secrétariat mathématique, Paris, 1962.

[Har95] Joe Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course, Corrected reprint of the 1992 original.

[Ive70] Birger Iversen. *Linear determinants with applications to the Picard scheme of a family of algebraic curves*, volume Vol. 174 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970.

[Mum08] David Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics*. Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition.

[Nak99] Hiraku Nakajima. *Lectures on Hilbert schemes of points on surfaces*, volume 18 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[Ser84] Jean-Pierre Serre. *Groupes algébriques et corps de classes*, volume 7 of *Publications de l'Institut Mathématique de l'Université de Nancago [Publications of the Mathematical Institute of the University of Nancago]*. Hermann, Paris, second edition, 1984. Actualités Scientifiques et Industrielles, 1264. [Current Scientific and Industrial Topics].

[Sta25] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2025.

[Tsi14] Jacob Tsimerman. Introduction to Etale Cohomology, 2014.

Avraham Aizenbud, Faculty of Mathematical Sciences, Weizmann Institute of Science, 76100 Rehovot, Israel
Email address: aizenr@gmail.com
URL: https://www.wisdom.weizmann.ac.il/~aizenr/

Dmitry Gourevitch, Faculty of Mathematical Sciences, Weizmann Institute of Science, 76100 Rehovot, Israel
Email address: dimagur@weizmann.ac.il
URL: https://www.wisdom.weizmann.ac.il/~dimagur/

David Kazhdan, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givaat Ram The Hebrew University of Jerusalem, Jerusalem, 91904, Israel
Email address: david.kazhdan@mail.huji.ac.il
URL: https://math.huji.ac.il/~kazhdan/

Eitan Sayag, Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, Be'er Sheva 84105, ISRAEL
Email address: eitan.sayag@gmail.com
URL: www.math.bgu.ac.il/~sayage