EXERCISE 1 IN ALGEBRAIC NUMBER THEORY

A remark on different kinds of problems. In all my home assignments I will use the following system. The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me). The problems marked by (P) you should hand in for grading. The sign (*) marks more difficult problems. The sign (□) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

This homework is composed from problems to section 1 in the books by Neukirch and Marcus.

(1) Show that the ring $\mathbb{Z}[i]$ cannot be ordered.

(2) (P) Let $d > 1 \in \mathbb{Z}$. Show that the only units (*i.e.* invertible elements) in $\mathbb{Z}[\sqrt{-d}]$ are ± 1, while the ring $\mathbb{Z}[\sqrt{d}]$ has infinitely many units.

(3) (P) Show that the ring $\mathbb{Z}[\sqrt{2}]$ is Euclidean and the units are given by $\pm(1+\sqrt{2})^n$, $n \in \mathbb{Z}$. Determine all the prime elements of this ring.

(4) Let $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$.
   (a) Show that $\mathbb{Z}[\omega]$ is spanned over $\mathbb{Z}$ by 1 and ω, and is preserved under complex conjugation.
   (b) Define a norm on $\mathbb{Z}[\omega]$ by $N(x) = x\overline{x}$. Show that $N(a+b\omega) = a^2 - ab + b^2 \in \mathbb{Z}$.
   (c) Show that $x \in \mathbb{Z}[\omega]$ is a unit if and only if $N(x) = 1$, and find all the units.
   (d) Show that $\mathbb{Z}[\omega]$ is a principal ideal domain.
   (e) Show that 2 is prime in $\mathbb{Z}[\omega]$, while $3 = u(1-\omega)^2$, where u is a unit.
   (f) (*) Give a description of all the primes of $\mathbb{Z}[\omega]$.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgNumTh.html

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