EXERCISE 9 IN ALGEBRAIC NUMBER THEORY

(1) (P) The field $\mathbb{Q}_p$ of p-adic numbers has no non-trivial automorphisms.

(2) (P) (i) The sequence $1,\ 1/10,\ 1/10^2,\ \cdots$ does not converge in $\mathbb{Q}_p$, for any p.
(P) (ii) For every $a \in \mathbb{Z}$, $(a, p) = 1$, the sequence $\{a^{p^n}\}_{n\in\mathbb{N}}$ converges in $\mathbb{Q}_p$.

(3) (P^*) Let $\epsilon \in 1 + p\mathbb{Z}_p$, and let $\alpha = a_0 + a_1 p + a_2 p^2 \cdots$ be a p-adic integer. Let $s_n = a_0 + a_1 p + \cdots + a_{n-1} p^{n-1}$. Show that the sequence ϵ^{s_n} converges to a number ϵ^α in $1 + p\mathbb{Z}_p$. Show that this turns $1 + p\mathbb{Z}_p$ into a multiplicative $\mathbb{Z}_p$-module.

(4) (P) The fields $\mathbb{Q}_p$ and $\mathbb{Q}_q$ are not isomorphic, unless $p = q$.

(5) (P^*) The algebraic closure of $\mathbb{Q}_p$ has infinite degree.