EXERCISE 11 IN ALGEBRAIC NUMBER THEORY

(1) (P) Show that the absolute values of the field $\mathbb{Q}(\sqrt{5})$ are given, up to equivalence, as follows.

   (i) $|a+b\sqrt{5}|_1 = |a+b\sqrt{5}|$ and $|a+b\sqrt{5}|_2 = |a-b\sqrt{5}|$ are the archimedean absolute values.

   (ii) If $p = 2$ or 5 or a prime $\neq 2, 5$ such that $(\frac{p}{5}) = -1$, then there is exactly one extension of $|\ \ |_p$ to $\mathbb{Q}(\sqrt{5})$, namely
$$|a+b\sqrt{5}|_p = |a^2-5b^2|_p^{1/2}.$$

   (iii) If p is a prime number $\neq 2, 5$ such that $(\frac{p}{5}) = 1$, then there are two extensions of $|\ \ |_p$ to $\mathbb{Q}(\sqrt{5})$, namely
$$|a+b\sqrt{5}|_{\mathfrak{p}_1} = |a+b\gamma|_p, \quad \text{resp.} \quad |a+b\sqrt{5}|_{\mathfrak{p}_2} = |a-b\gamma|_p,$$
   where γ is a solution of $x^2-5=0$ in $\mathbb{Q}_p$.

(2) (i) Show that $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{Q}_5$.

   (ii) Find all the solutions of $x^2 = 14 \mod 625$.

   (iii) How many roots does $f(x) = x^3 - x - 2$ have $\mod 2^{10}$?

(3) (P^*) Show that for a $\mathfrak{p}$-adic field K, every subgroup of finite index in K^* is both open and closed.

(4) (P^*) Show that the maximal unramified extension of $\mathbb{Q}_p$ is obtained by attaching all the roots of unity of order prime to p. Notice that this infinite extension still has discrete valuation. See Proposition (7.12).

(5) (P^*) Let ζ be a primitive p^m-th root of unity. Then one has:

   (i) $\mathbb{Q}_p(\zeta)/\mathbb{Q}_p$ is totally ramified of degree $\phi(p^m)$, where ϕ is the Euler totient function.

   (ii) $G(\mathbb{Q}_p(\zeta)/\mathbb{Q}_p) \simeq (\mathbb{Z}/p^m\mathbb{Z})^\times$.

   (iii) $\mathbb{Z}_p[\zeta]$ is the valuation ring of $\mathbb{Q}_p(\zeta)$.

   (iv) $1-\zeta$ is a prime element of $\mathbb{Z}_p[\zeta]$ with norm p.

   (v) Let $\mathfrak{p}$ be the unique maximal ideal. What is the image of $v_{\mathfrak{p}}(Z_p[\zeta])$?

(6) (P^*) Using questions (4) and (5) above, what can you say about the structure of $G(\mathbb{Q}_p^{ab}/\mathbb{Q}_p)$, where $\mathbb{Q}_p^{ab}$ denotes the maximal abelian extension of $\mathbb{Q}_p$ in its algebraic closure?