EXERCISE 2 IN ALGEBRAIC NUMBER THEORY

A remark on different kinds of problems. In all my home assignments I will use the following system. The problems without marking are just exercises. You have to convince yourself that you can do them but it is not necessary to write them down (if you have difficulties with one of these problems ask me). The problems marked by (P) you should hand in for grading. The sign (*) marks more difficult problems. The sign (□) marks more challenging and more interesting problems which are related to some interesting subjects. They are not always related to the course material, but I definitely advise you to think about these problems.

This homework is consists of problems to section 2 in the book of Neukirch.

(1) (P) Is $\frac{3+\sqrt{6}}{1-\sqrt{6}}$ an algebraic integer?

(2) (P) Let D be a square-free rational integer, $D \notin \{0,1\}$ and d be the discriminant of the quadratic number field $K = \mathbb{Q}(\sqrt{D})$. Show that $d = \begin{cases} D, & \text{if } D \equiv 1 \bmod 4 \\ 4D, & \text{if } D \equiv 2 \text{ or } 3 \bmod 4 \end{cases}$,
and that an integral basis of K is given by $\{1, \sqrt{D}\}$ in the second case, by $\{1, \frac{1}{2}(1+\sqrt{D})\}$ in the first case, and by $\{1, \frac{1}{2}(d+\sqrt{d})\}$ in both cases.

(3) Show that $\{1, \sqrt[3]{2}, \sqrt[3]{2}^2\}$ is an integral basis of $\mathbb{Q}(\sqrt[3]{2})$.

(4) Show that $\{1, \theta, \frac{1}{2}(\theta+\theta^2)\}$ is an integral basis of $\mathbb{Q}(\theta)$, $\theta^3 - \theta - 4 = 0$.

(5) (*) The discriminant d_K of an algebraic number field K is always either divisible by 4 or 1 mod 4. (See the book for a hint).

URL: http://www.wisdom.weizmann.ac.il/~dimagur/AlgNumTh.html

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