EXERCISE 3 IN ALGEBRAIC NUMBER THEORY

(1) Show that
$$54 = 2 \cdot 3^3 = \frac{13+\sqrt{-47}}{2} \cdot \frac{13-\sqrt{-47}}{2}$$
are two essentially different decompositions into irreducible integral elements of $\mathbb{Q}(\sqrt{-47})$.

(2) (P) Let d be squarefree and p a prime number not dividing $2d$. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{d})$. Show that $(p) = p\,\mathcal{O}$ is a prime ideal if and only if the congruence $x^2 \equiv d \mod p$ has no solution.

(3) (P) The quotient ring $\mathcal{O}/\mathfrak{a}$ of a Dedekind domain by an ideal $\mathfrak{a} \neq 0$ is principal. (See book for a hint).

(4) (P) Every ideal of a Dedekind domain can be generated by two elements. (Use Ex. 3).

(5) (P)* Let $\mathfrak{m}$ be a nonzero integral ideal of the Dedekind domain $\mathcal{O}$. Let K denote the fraction field of $\mathcal{O}$. Show that in every ideal class of the class group Cl_K, there exists an integral ideal prime to $\mathfrak{m}$.

(6) (P) The fractional ideals $\mathfrak{a}$ of a Dedekind domain $\mathcal{O}$ are projective $\mathcal{O}$-modules, i.e., given any surjective homomorphism $M \xrightarrow{f} N$ of $\mathcal{O}$-modules, each homomorphism $\mathfrak{a} \xrightarrow{g} N$ can be lifted to a homomorphism $h : \mathfrak{a} \to M$ such that $f \circ h = g$.