EXERCISE 4 IN ALGEBRAIC NUMBER THEORY

(1) (P) Show that a lattice Γ in $\mathbb{R}^n$ is complete if and only if the quotient $\mathbb{R}^n/\Gamma$ is compact.

(2) (P) Show that Minkowski's lattice point theorem (LPT) cannot be improved, by giving an example of a centrally symmetric convex set $X \subset V$ such that $\mathrm{vol}(X) = 2^n \mathrm{vol}(\Gamma)$ which does not contain any nonzero point of the lattice Γ. Further, if X is compact, the statement of LPT remains true in the case of equality as well.

(3) (P*) Let K be a number field and let O_K denote its ring of integers. Show that in every ideal $\mathfrak{a} \neq 0$ of O_K there exist an element $a \neq 0 \in \mathfrak{a}$ such that

$$|N_{K/\mathbb{Q}}(a)| \leq M(O_K : \mathfrak{a}),$$

where $M = \frac{n!}{n^n} \cdot (\frac{4}{\pi})^s \sqrt{|d_K|}$ (the so-called Minkowski bound). See the book for a hint.

(4) Find a constant A which depends only on K such that any integral ideal $\mathfrak{a} \neq 0$ of K contains an element $a \neq 0$ satisfying

$$|\tau a| < A(O_K : \mathfrak{a})^{1/n} \text{ for all } \tau \in \mathrm{Hom}(K, \mathbb{C}), \text{ where } n = [K : Q].$$