EXERCISE 5 IN ALGEBRAIC NUMBER THEORY

(1) (P) Show that the quadratic fields with discriminant 5, 8, 11, -3, -4, -7, -8, -11 have class number 1. Is the class number of $\mathbb{Q}(\sqrt{-23}) = 1$?

(2) (P) Let d_K denote the discriminant of the number field K. Show that the absolute value $|d_K| > 1$.
Hint: Use Minskowski's bound from assignment 4.

(3) (P*) Let $\mathfrak{a}$ be an integral ideal of K such that $\mathfrak{a}^n$ is a principal ideal generated by $a \in \mathcal{O}_K$, say. Show that $\mathfrak{a}\mathcal{O}_L$ is a principal ideal in the extension field $L = K(\sqrt[n]{a})$. Furthermore, show that for every number field K, there exists a finite extension L such that every ideal of K becomes a principal ideal in L.

(4) (P) Describe the group of units in $\mathbb{Q}(\sqrt{D})$, where $D \in \{1, -1, 2, 3, -3, 5, 6, 10\}$.

(5) (P) Let ζ be the 5-th root of unity. Show that $\mathbb{Z}[\zeta]^* = \{\pm\zeta^k(1+\zeta)^n | 0 \leq k < 5,\ n \in \mathbb{Z}\}$.