EXERCISE 8 IN ALGEBRAIC NUMBER THEORY

(1) For every finite abelian group A there exists a Galois extension $L/\mathbb{Q}$ with Galois group $G(L/\mathbb{Q}) \simeq A$.
Hint: Use the cyclotomic extension.

(2) (P) Every quadratic number field $\mathbb{Q}(\sqrt{d})$ is contained in some cyclotomic field $\mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n-th root of unity. Show that $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{-2})$ are the quadratic subfields of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, for $n = 2^q$, $q \geq 3$.

(3) (P) Describe the quadratic subfields of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, in the case where n is a product of two odd primes.

(4) Let $f : A \to B$ be a homomorphism of integral domains and S a multiplicatively closed subset such that $f(S) \subseteq B^\times$. Then f induces a homomorphism $A_S \to B$.

(5) (P) Let A be an integral domain. If the localization A_S is integral over A, then $A_S = A$.

For the remaining exercises we let A be an arbitrary ring (commutatve, with 1) but not necessarily an integral domain. Let M be an A-module and S be a multiplicatively closed subset of A such that $0 \notin S$. In $M \times S$, consider the equivalence relation

$$(m, s)\ (m', s') \iff \exists s'' \in S \text{ such that } s''(s'm - sm') = 0.$$

Show that the set M_S of equivalence classes $\overline{(m, s)}$ forms an A-module, and that $M \to M_S$, $a \mapsto \overline{(a, 1)}$, is a homomorphism . In particular, A_S is a ring. It is called the **localization** of A with respect to S.

(6) (P) Show that localization is an exact functor. In other words, for any exact sequence $L \to M \to N$, the sequence $L_S \to M_S \to N_S$ is also exact.

(7) (P) Show that for any A-module M, the following conditions are equivalent:
(i) $M = 0$.
(ii) $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$.

(iii) $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$.

(8) (P) Let A be an integral domain and let $f : M \to N$ be a homomorphism of A-modules. Then the following conditions are equivalent:
  (i) f is injective (surjective).
  (ii) $f_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective (surjective) for every prime ideal $\mathfrak{p}$.
  (iii) $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective (surjective) for every maximal ideal $\mathfrak{m}$.

(9) (**Nakayama's Lemma**) Let A be a local ring with maximal ideal $\mathfrak{m}$. Let M be an A-module and let $N \subseteq M$, a submodule such that M/N is finitely generated. Then
$$M = N + \mathfrak{m}M \Longrightarrow M = N.$$