EXERCISE 9 IN ALGEBRAIC NUMBER THEORY

(1) (P) Let K be a number field such that $[K:\mathbb{Q}] = n$. Define a subgroup M of K as
$$M = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n,$$
where $\alpha_1, \cdots, \alpha_n$ form a basis of $K/\mathbb{Q}$. Show that the ring of multipliers
$$\mathfrak{o} = \{\alpha \in K | \alpha M \subseteq M\}$$
is an order in K, but in general not the maximal order.

(2) (P) In an order $\mathfrak{o}$ of K, the following are equivalent;

(i) $\mathfrak{p}$ is a non-zero regular prime ideal

(ii) $\mathfrak{p}$ is invertible

(iii) the set $\{x \in K | x\mathfrak{p} \subseteq \mathfrak{p}\} = \mathfrak{o}$

(3) (P) Let $\mathfrak{a}$ be an $\mathcal{O}_K$-ideal of K. Show that $\mathfrak{o} = \mathbb{Z} + \mathfrak{a}\mathcal{O}_K$ is an order. Compute the conductor of $\mathfrak{o}$.

(4) (P^*) Recall that in a dedekind domain, every invertible ideal is generated by atmost **two** elements. Is it true for invertible ideals in an order as well?