On classification of hypergeometric orthogonal polynomials

Dmitry Gourevitch (Weizmann Institute, Israel)

Cable Car Algebra Seminar, Haifa
j.w. J. Bernstein & S. Sahi

http://www.wisdom.weizmann.ac.il/~dimagur/

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- Write $P_n = \sum_{k=0}^{n} c(n,k) x^k$. Say P_n is *of Jacobi type* if it is quasi-orthogonal and $\exists$ polynomials $p(u,s)$, $w(s)$ s.t.

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Equivalence: $P_n(x) \sim P_n(\lambda x)$, $P_n(x) \sim e_n P_n(x)$, $e_n \in \mathbb{C}^{\times}$.

Theorem (Bernstein-G.-Sahi '24)

There exist only five families of Jacobi type (up to $\sim$): Jacobi, Laguerre, Bessel, and two families E_n, F_n obtained from Lommel polynomials.

Definitions of the families

$$ {}_iF_j(\underline{a};\underline{b};x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_i)_k}{(b_1)_k \cdots (b_j)_k k!} x^k, \quad (c)_k := c(c+1)\cdots(c+k-1), $$

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$$ (D(D-1/2) - x(D-n)(D-n-c+1)(D+n+c)(D+n+1))E_n^{(c)} = 0 $$

$$ (D(D+1/2) - x(D-n)(D-n-c+2)(D+n+c)(D+n+2))F_n^{(c)} = 0 $$

where $D := x\frac{d}{d_x}$

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- $h_{2n}^{(c)} = (-1)^n E_n^{(c)}(-x^2), \quad h_{2n+1}^{(c)} = (-1)^n x F_n^{(c-1)}(-x^2),$
- $h_{n+1}^{(c)} = 2(c+n)xh_n^{(c)} - h_{n-1}^{(c)}, \quad h_0 = 1,\ h_{-1} = 0.$

Theorem (Bernstein-G.-Sahi '24)

If $\{P_n\}_{n=0}^{\infty}$ *is a quasi-orthogonal family with*

$$\frac{c(n,k+1)}{c(n,k)} = f(n,k) \text{ for some rational function } f \in \mathbb{C}(u,s)$$

then there exists $g \in \mathbb{C}[u,s]$ *and a family* $\{Q_n\}_{n=0}^{\infty}$ *such that*

$$P_n = g(n, x\partial_x)Q_n \quad \forall n,$$

and $\{Q_n\}$ *is either Jacobi, or Laguerre, or Bessel, or*
$Q_n(x) = {}_4F_1(-n, -n+d, n+a, n+c; b; x)$
for some scalars $a, b, c, d \in \mathbb{C}$.

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Example

1. $P_n = {}_3F_2(-n, n+1, cn+\frac{c+3}{2}; 3/2, cn+\frac{c+1}{2}; x)$,
   $Q_n = Jacobi(1, 3/2)$.
2. Families not of the form ${}_iF_j$.

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Question

1. *Can it be that Q_n is always of Jacobi type?*
2. *Given a Q_n of Jacobi type, what P_n are possible?*

Families of HG type

Generalization: let $R(s) \in \mathbb{C}[s]$, and define a family Φ_k of monic polynomials by

$$\Phi_{-1} = 0,\ \Phi_0 = 1,\ \Phi_{k+1} = (z + R(k))\Phi_k$$

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Definition

Let $\{P_n\}$ be a quasi-orthogonal family of polynomials. We say that it is of **rational HG type** *if there exists a rational function $f(u,s)$, and a polynomial $R(s)$, such that for any $n \in \mathbb{Z}_{\geq 0}$ we have $P_n = \sum_{k=0}^{n} c(n,k)\Phi_k$ where $c(n,k)$ satisfy*

$$c(n,k+1) = f(n,k)c(n,k) \quad \text{for all } n,k \in \mathbb{Z}. \tag{1}$$

We say that $\{P_n\}$ is of **HG type** *if it is of rational HG type, and the denominator of $f(u,s)$ does not depend on u.*

Theorem (Bernstein-G.-Sahi '25)

Any family $\{P_n\}$ *of HG type arises by a rescaling* $P_n(z) \mapsto P_n(ez)$ *and/or a renormalization* $P_n(z) \mapsto e_n P_n(z)$ *and/or shift* $P_n(z) \mapsto P_n(z+e)$ *from a family given by a pair* (f, R) *such that* $f(u,s) = \frac{s-u}{s+1} f_1(u,s)$, $R(s) = sr(s-1)$, *and* $(f_1(u,s), r(s))$ *belongs to the following list.*

(a) $r(s) \in \{0, 1, s+a\}$ *and* $f_1(u,s) = (sr(s) + bs + d)^{-1}$, *for* $a, b, d \in \mathbb{C}$

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$$f_1(u,s) = \frac{u+s+c}{w(s)},$$
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- (c) $r(s) \equiv 0$ *and for some* $c \in \mathbb{C}$ *we have either* $$f_1(u,s) = \frac{(s-u-c+1)(u+s+1)(u+s+c)}{s+1/2} \text{ or}$$ $$f_1(u,s) = \frac{(s-u-c+2)(u+s+2)(u+s+c)}{s+3/2}$$

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(a) $\{Q_n\}$ *is of HG type*

(b) $\{Q_n\}$ *is given by* $(f_1(u,s), 0)$ *where*

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Finite families are obtained from HG type families by substituting special parameters.

Proof Ingredients

- Gauss-Favard: $\{P_n\}$ monic quasi-orthogonal $\iff \exists\{\alpha_n\},\{\beta_n\} \in \mathbb{C}^{\mathbb{N}}$

$$\text{s.t. } xP_n = P_{n+1} + \alpha_n P_n + \beta_n P_{n-1}, \quad \beta_n \neq 0 \text{ for all } n \geq 1; \qquad (2)$$

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- Algebra $A=\mathbb{C}(u,s)\langle U^{\pm 1}, S^{\pm 1}\rangle$ with $Uu=u+1, Ss=s+1$.
- Key lemma: If $\{P_n\}$ of HG type then $\alpha_n,\beta_n\in\mathbb{C}(n)$ (almost), and $c(n,k)$ generate an A-module that is 1-dimensional over $\mathbb{C}(u,s)$.

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- Finish, using clever but elementary considerations.

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Thank you for your attention!