On classification of hypergeometric orthogonal polynomials

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Orthogonal families of hypergeometric polynomials

- Let $\{P_n\}_{n=0}^{\infty}$ be a family of polynomials in $\mathbb{C}[x]$ with $\deg P_n = n$.
- $\{P_n\}$ is a *quasi-orthogonal* family if there exists a linear functional $M : \mathbb{C}[x] \to \mathbb{C}$ s.t. $M(P_i P_j) = 0 \iff i \neq j$.
- Write $P_n = \sum_{k=0}^{n} c(n,k)x^k$. Say P_n is *of Jacobi type* if it is quasi-orthogonal and $\exists$ polynomials $p(u,s)$, $w(s)$ s.t.

$$\frac{c(n,k+1)}{c(n,k)} = \frac{p(n,k)}{w(k)}$$

Equivalence: $P_n(x) \sim P_n(\lambda x)$, $P_n(x) \sim e_n P_n(x)$, $e_n \in \mathbb{C}^{\times}$.

Theorem (Bernstein-G.-Sahi '24)

There exist only five families of Jacobi type (up to $\sim$): Jacobi, Laguerre, Bessel, and two families E_n, F_n obtained from Lommel polynomials.

Definitions of the families

$$ {}_iF_j(\underline{a};\underline{b};x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_i)_k}{(b_1)_k \cdots (b_j)_k k!} x^k, \quad (c)_k := c(c+1)\cdots(c+k-1), $$

If $a_1 = -n$ the infinite series truncates to a polynomial of degree $\leq n$.

1. Jacobi:${}_2F_1(-n, n+a; b; x)$ measure:$(1-x)^{b-1}(1+x)^{a-b}dx$ on $[0,1]$
   $p(u,s) = (s-u)(s+u+a), \quad w(s) = (s+1)(s+b)$.
2. Bessel:${}_2F_0(-n, n+a; ; x)$ measure: $x^{a-1}\exp(1/x)dx$ on $(-\infty, 0)$
3. Laguerre: ${}_1F_1(-n; b; x)$ measure: $x^{b-1}\exp(-x)dx$ on $(0, \infty)$
4. $E_n^{(c)}$: ${}_4F_1(-n, -n-c+1, n+c, n+1; 1/2; x)$
5. $F_n^{(c)}$: ${}_4F_1(-n, -n-c+2, n+c, n+2; 3/2; x)$

Measures of E_n, F_n are discrete measures defined using zeros of (modified) Bessel functions.

$$ (D(D-1/2) - x(D-n)(D-n-c+1)(D+n+c)(D+n+1))E_n^{(c)} = 0 $$

$$ (D(D+1/2) - x(D-n)(D-n-c+2)(D+n+c)(D+n+2))F_n^{(c)} = 0 $$

where $D := x\frac{d}{d_x}$

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- $h_{2n}^{(c)} = (-1)^n E_n^{(c)}(-x^2), \quad h_{2n+1}^{(c)} = (-1)^n x F_n^{(c-1)}(-x^2),$
- $h_{n+1}^{(c)} = 2(c+n)xh_n^{(c)} - h_{n-1}^{(c)}, \quad h_0 = 1, \ h_{-1} = 0.$

Theorem (Bernstein-G.-Sahi '24)

If $\{P_n\}_{n=0}^{\infty}$ *is a quasi-orthogonal family with*

$$\frac{c(n,k+1)}{c(n,k)} = f(n,k) \text{ for some rational function } f \in \mathbb{C}(u,s)$$

then there exists $g \in \mathbb{C}[u,s]$ *and a family* $\{Q_n\}_{n=0}^{\infty}$ *such that*

$$P_n = g(n, x\partial_x)Q_n \quad \forall n,$$

and $\{Q_n\}$ *is either Jacobi, or Laguerre, or Bessel, or*
$Q_n(x) = {}_4F_1(-n, -n+d, n+a, n+c; b; x)$
for some scalars $a, b, c, d \in \mathbb{C}$.

Example

1. $P_n = {}_3F_2(-n, n+1, cn + \frac{c+3}{2}; 3/2, cn + \frac{c+1}{2}; x)$,
   $Q_n = Jacobi(1, 3/2)$.
2. Families not of the form ${}_iF_j$.

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Question

1. *Can it be that* Q_n *is always of Jacobi type?*
2. *Given a* Q_n *of Jacobi type, what* P_n *are possible?*

Families of HG type

Generalization: let $R(s) \in \mathbb{C}[s]$, and define a family Φ_k of monic polynomials by

$$\Phi_{-1} = 0,\ \Phi_0 = 1,\ \Phi_{k+1} = (z + R(k))\Phi_k$$

Definition

Let $\{P_n\}$ be a quasi-orthogonal family of polynomials. We say that it is of **rational HG type** *if there exists a rational function $f(u,s)$, and a polynomial $R(s)$, such that for any $n \in \mathbb{Z}_{\geq 0}$ we have $P_n = \sum_{k=0}^{n} c(n,k)\Phi_k$ where $c(n,k)$ satisfy*

$$c(n, k+1) = f(n,k)c(n,k) \quad \textit{for all } n, k \in \mathbb{Z}. \tag{1}$$

We say that $\{P_n\}$ is of **HG type** *if it is of rational HG type, and the denominator of $f(u,s)$ does not depend on u.*

Theorem (Bernstein-G.-Sahi '25)

Any family $\{P_n\}$ *of HG type arises by a rescaling* $P_n(z) \mapsto P_n(ez)$ *and/or a renormalization* $P_n(z) \mapsto e_n P_n(z)$ *and/or shift* $P_n(z) \mapsto P_n(z+e)$ *from a family given by a pair* (f, R) *such that* $f(u,s) = \frac{s-u}{s+1} f_1(u,s)$*,* $R(s) = sr(s-1)$*, and* $(f_1(u,s), r(s))$ *belongs to the following list.*

- (a) $r(s) \in \{0, 1, s+a\}$ *and* $f_1(u,s) = (sr(s) + bs + d)^{-1}$*, for* $a, b, d \in \mathbb{C}$
- (b)
$$f_1(u,s) = \frac{u+s+c}{w(s)},$$
and $r(s)$ *is the quotient obtained by Euclidean division of* $w(s)$ *by* $s(s+c)$*. Here,* $w(s)$ *is a monic polynomial of degree* ≤ 3 *and* $c \in \mathbb{C}$*.*
- (c) $r(s) \equiv 0$ *and for some* $c \in \mathbb{C}$ *we have either*
$$f_1(u,s) = \frac{(s-u-c+1)(u+s+1)(u+s+c)}{s+1/2} \text{ or}$$
$$f_1(u,s) = \frac{(s-u-c+2)(u+s+2)(u+s+c)}{s+3/2}$$

Classically, families in type (a) are called Laguerre, Charlier, Meixner, and continuous dual Hahn, and in type (b) Bessel, Jacobi, continuous Hahn, and Wilson.

Theorem (Bernstein-G.-Sahi '25)

For any family $\{P_n\}$ of rational HG type there exists $g(u,s) \in \mathbb{C}(s)[u]$ and a family $Q_n = c'(n,k)\Phi_k$ such that $P_n(z) = \sum g(n,k)c'(n,k)\Phi_k$, and one of the following holds

- (a) $\{Q_n\}$ *is of HG type*
- (b) $\{Q_n\}$ *is given by* $(f_1(u,s), 0)$ *where*

$$f_1(u,s) = \frac{(s-u-b)q(u+s)}{s+d},$$

for some monic quadratic polynomial $q \in \mathbb{C}[s]$ and scalars $b, d \in \mathbb{C}$.

Question: would you call $P_n = {}_3F_2(-n, n+1, cn + \frac{c+3}{2}; 3/2, cn + \frac{c+1}{2}; x)$ a hypergeometric orthogonal family?
Finite families are obtained from HG type families by substituting special parameters.

Proof Ingredients

- Gauss-Favard: $\{P_n\}$ monic quasi-orthogonal $\iff \exists\{\alpha_n\},\{\beta_n\}\in\mathbb{C}^{\mathbb{N}}$

$$\text{s.t. } xP_n = P_{n+1}+\alpha_n P_n+\beta_n P_{n-1}, \quad \beta_n\neq 0 \text{ for all } n\geq 1; \qquad (2)$$

- Algebra $A=\mathbb{C}(u,s)\langle U^{\pm1},S^{\pm1}\rangle$ with $Uu=u+1, Ss=s+1$.
- Key lemma: If $\{P_n\}$ of HG type then $\alpha_n,\beta_n\in\mathbb{C}(n)$ (almost), and $c(n,k)$ generate an A-module that is 1-dimensional over $\mathbb{C}(u,s)$.
- Ore (1930s): models for all 1-dimensional A-modules using products of functions of the form $\Gamma(ku+ls)$, $k,l\in\mathbb{Z}$ and $\exp(au+bs)$.
- Elementary analysis of poles using these models and (2):

$$\Psi(u,s-1)=\Psi(u+1,s)+(\alpha(u)+R(s))\Psi(u,s)+\beta(u)\Psi(u-1,s),$$

$$\text{where } \Psi = g\exp(au+bs)\prod\Gamma(k_iu+l_is+c_i),\ g\in\mathbb{C}(u,s)\Rightarrow$$

$$(k_i,l_i)\in\{(\pm1,1),(0,-1),(\pm1,0)\}\ \forall i\Rightarrow$$

$$\Psi(u,s+1)=\frac{p(s-u)q(s+u)Sg}{w(s)g}\Psi(u,s)$$

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$$\text{where } \Psi = g\exp(au+bs)\prod\Gamma(k_i u + l_i s + c_i), \; g \in \mathbb{C}(u,s) \Rightarrow$$

$$(k_i, l_i) \in \{(\pm 1, 1), (0,-1), (\pm 1, 0)\} \; \forall i \Rightarrow$$

$$\Psi(u,s+1) = \frac{p(s-u)q(s+u)Sg}{w(s)g}\Psi(u,s)$$

- Finish, using clever but elementary considerations.

Future plans

- basic HG families (q-hypergeometric) - in progress.
- rational HG families.

Askey-Wilson, 1979: A characterization theorem that leads to new orthogonal polynomials is usually interesting, one that says the classical polynomials are the only polynomials with a given property is usually much less interesting and if it keeps people from looking for new polynomials it is harmful.

Fortunately, ours found new polynomials, and leaves open questions.

Thank you for your attention!