Perverse Sheaves - Exercise 1

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Remarks on different types of problems: problems without any marks are easy. You do not have to write the solution. You can still do so if you wish us to check it. Problems marked (P) are usual problems for submission. Problems marked (*) are difficult.

Notation: For a topological space X, we denote by $Sh(X)$, resp. $PSh(X)$, the categories of sheaves, resp. presheaves, of sets, and by $Ab(X)$, resp. $PAb(X)$, the categories of sheaves, resp. presheaves, of abelian groups.

1. (a) (P) Does the category $PSh(X)$ have initial and finite objects? Does it have limits? Colimits?
   (b) (P) The same question for the category $Sh(X)$.

2. (P) Let $\nu : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets on X.
   (a) Show that it is injective (resp. surjective, resp. isomorphism) if and only if for any point $p \in X$ the corresponding morphism of stalks $\nu_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective (resp. surjective, resp. isomorphism)?
   (b) Are similar statements correct for sections? Namely,
      i. Is injectivity of $\mathcal{F} \to \mathcal{G}$ equivalent to injectivity of $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open subset $U \subset X$?
      ii. Is surjectivity of $\mathcal{F} \to \mathcal{G}$ equivalent to surjectivity of $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open subset $U \subset X$?

3. Let $\pi : X \to Y$ be a continuous map. Show that
   (a) the functor $\pi_\cdot$ preserves all small limits in $Sh(X)$.
   (b) the functor $\pi^\cdot$ preserves all small colimits in $Sh(Y)$.
   (c) (*) the functor $\pi^\cdot$ preserves all finite limits in $Sh(Y)$, but not always preserves infinite limits.
   (d) the functor $\pi_\cdot$ is strictly left exact on the category $Ab(X)$.
   (e) the functor $\pi^\cdot$ is strictly right exact on the category $Ab(X)$.
   (f) (*) the functor $\pi^\cdot$ is left exact on the category $Ab(X)$, but not always strictly left exact on the category $Ab(X)$.

4. (a) (P) Show that the category $PAb(X)$ is abelian. Describe kernel, cokernel, image in this category.
   (b) (P) Show that the category $Ab(X)$ is abelian. Describe kernel, cokernel, image in this category.

5. (P) Show that

   (a) A sequence of presheaves of abelian groups is exact if and only if for any open subset U, the corr. sequence of sections on U is exact.

   (b) A sequence of sheaves of abelian groups is exact if and only if for every point $p \in X$, the corr. sequence of stalks at p is exact.

6. (P) For each of the following functors, determine whether it has a left adjoint, and whether it has a right adjoint. Describe the adjoints it has. Here, Ab denotes the category of abelian groups, Gr the category of groups, Top denotes the category of topological spaces, For denotes forgetful functors into sets.

   (a) $For : Ab \to Sets$

   (b) $For : Top \to Sets$

   (c) The inclusion $Ab \to Gr$

   (d) The inclusion $Gr \to Sets$

   (e) The inclusion $Sh(X) \to PSh(X)$.

   (f) The inclusion $Ab(X) \to PAb(X)$.