Perverse Sheaves - Exercise 2: Adjoint functors

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April 29, 2022

Remarks on different types of problems: problems without any marks are easy. You do not have to write the solution. You can still do so if you wish us to check it. Problems marked (P) are usual problems for submission. Problems marked (*) are difficult.

Notation: For a topological space X, we denote by $Sh(X)$, resp. $PSh(X)$, the categories of sheaves, resp. presheaves, of sets, and by $Ab(X)$, resp. $PAb(X)$, the categories of sheaves, resp. presheaves, of abelian groups.

1. (P) Prove the Yoneda lemma: an object is uniquely defined by the sets of it morphisms to other objects (including itself). More preciselty, if for two objects, the sets of morphisms into any (third) object in the category are functorially bijective, then the two objects are canonically isomorphic.

   In more detail, let $\mathcal{C}$ be a category, and let $X, Y \in Ob(\mathcal{C})$. Suppose that for any $Z \in Ob(\mathcal{C})$, we are given a bijection $\Psi_Z : Mor(X,Z) \simeq Mor(Y,Z)$, such that for any morphism $\varphi : Z \to Z'$, the following diagram

   $$\begin{array}{ccc} Mor(X,Z) & \xrightarrow{\Psi_Z} & Mor(Y,Z) \\ {\scriptstyle \varphi\circ}\downarrow & & \downarrow{\scriptstyle \varphi\circ} \\ Mor(X,Z') & \xrightarrow{\Psi_{Z'}} & Mor(Y,Z'). \end{array}$$

   is commutative. Construct an isomorphism $X \simeq Y$.

   In the same way one proves a contravariant version of this lemma, in which $\forall Z \in Ob(\mathcal{C})$ one has a functorial bijection $Mor(Z,X) \simeq Mor(Z,Y)$. This version is equivalent to the previous one applied to the opposite category $\mathcal{C}^{op}$.

2. Show that the left (resp. right) adjoint functor (if exists) is defined uniquely up to unique isomorphism.

3. (P) Let $\mathcal{C}, \mathcal{D}$ be categories, and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be functors between them.

   (i) Suppose $\mathcal{F}$ is left adjoint to $\mathcal{G}$. Show that we can canonically construct two adjunction morphisms

   $$i : \mathrm{Id}_{\mathcal{C}} \to \mathcal{G}\mathcal{F}, \quad j : \mathcal{F}\mathcal{G} \to \mathrm{Id}_{\mathcal{D}}\,.$$

   satisfying the following conditions

(i) $\mathcal{F} \to \mathcal{F}\mathcal{G}\mathcal{F} \to \mathcal{F}$ is Id

(ii) $\mathcal{G} \to \mathcal{G}\mathcal{F}\mathcal{G} \to \mathcal{G}$ is Id

(ii) Conversely, a pair of morphisms of functors i, j satisfying conditions (i) and (ii) defines an adjunction between $\mathcal{F}$ and $\mathcal{G}$.

(iii) Suppose we constructed a morphism of functors $i : \mathrm{Id}_{\mathcal{C}} \to \mathcal{G}\mathcal{F}$. We can suspect that this is an adjunction morphism for some adjunction between $\mathcal{F}$ and $\mathcal{G}$. Show that if this is the case, then the second adjunction morphism j is uniquely defined.

4. (P) For each of the following functors, determine whether it has a left adjoint, and whether it has a right adjoint. Describe the adjoints it has. Here, Ab denotes the category of abelian groups, Gr the category of groups, Top denotes the category of topological spaces, For denotes forgetful functors into sets.

(a) for a morphism $A \to B$ of algebras, the natural functor of restriction of scalars $Res : M(B) \to M(A)$ from B-modules to A-modules.

(b) For a field K and a vector space $W \in Vect_K$,

i. the tensor product functor $T_W : Vect_K \to Vect_K$ given by $T_W(V) = W \otimes V$.

ii. The functor $H_W : Vect_K \to Vect_K$ given by $H_W(V) := Hom(W, V)$ - the space of all linear operators $V \to W$.

(c) The inclusion $Sh(X) \to PSh(X)$.

(d) The inclusion $Ab(X) \to PAb(X)$.

(e) The inclusion of the category of complete normed spaces into the category of normed spaces.