EXERCISE 3 IN PERVERSE SHEAVES

SHACHAR CARMELI

For an abelian category $C$, let $K(C)$ denote the homotopy category of $C$ and $D(C)$ the derived category. If $f : A \to B$ we denote by $\text{Cone}(f)$ the mapping cone, and by $\pi_f : B \to \text{Cone}(f)$ the inclusion to the second factor.

1. Prove that $\text{Cone}(\pi_f) \cong A[1]$.

2. Prove that $H^i(A) \xrightarrow{f^*} H^i(B) \xrightarrow{(\pi_f)^*} H^i(\text{Cone}(f))$ is exact. Deduce that a distinguished triangle induces a long exact sequence on cohomologies.

3. (P) Prove that $f : A \to B$ is a homotopy equivalence if and only if $\text{Cone}(f)$ is contractible (i.e. isomorphic to 0 in $K(C)$).

4. Prove that $f : A \to B$ is a quasiisomorphism if and only if $\text{Cone}(f)$ is acyclic.

5. (*) Prove that every distinguished triangle in $K(C)$ is isomorphic to a short exact sequence. Deduce once again that a distinguished triangle induces a long exact sequence on cohomologies. Is every short exact sequence a distinguished triangle in $K(C)$?

6. (P) Prove that for $g : C \to B$ and $f : A \to B$, $g = f \circ g'$ if and only if $\pi_f \circ g = 0$. State and prove a dual statement.

7. (P) Define the complex $\text{Hom}(A, B)$, and prove that $H^0(\text{Hom}(A, B)) = \text{Hom}_{K(C)}(A, B)$. Prove that $\text{Hom}(X, C(f)) = C(\text{Hom}(X, f))$. Use this to solve the last exercise much faster!

8. (*) Prove that in $D^+(Ab)$ every object is isomorphic to a complex with zero differentials.

9. (P) Prove that if $I \in K^+(C)$ is a complex of injective objects, and $M \in K^+(C)$ is acyclic, then $\text{Hom}_{K(C)}(M, I) = 0$. Deduce that for every complex $A$, $\text{Hom}_{D^+(C)}(A, I) \cong \text{Hom}_{K^+(C)}(A, I)$.

10. Recall that $R\text{Hom}(A, B) := \text{Hom}(A, I)$ where $B \cong I$ in $D^+(C)$ and $I$ is a complex of injectives. Alternatively, if we work in $D^-(C)$ it is $\text{Hom}(P, A)$ where $A \cong P$ and $P$ is a complex of projectives. Prove that $\text{Hom}_{D^+(C)}(A, B) = H^0(R\text{Hom}(A, B))$, and similarly for $D^-(C)$.

11. (*) Find an example of a complex which is not quasi-isomorphic to a complex with zero differentials.

12. (P) Let $k$ be a field. Let $C = \text{Mod}(k[x, y])$. Let $A$ be the complex

$$k[x, y] \xrightarrow{x^2} k[x, y] \to k[x, y]/x$$
and \( B \) be the complex

\[
k[x, y]/(y^2 - x) \to k[x, y]/(y^2 - x) \to k[x, y]/(x, y).
\]

Compute the cohomologies of \( \mathcal{R} \text{Hom}(A, B) \) and \( A \otimes^L B \). Here, \( \mathcal{R} \text{Hom} \) is the derived functor of the \( \text{Hom} \) functor, and \( \otimes^L \) is the derived functor of the tensor product functor.

\textit{URL:} \url{http://www.wisdom.weizmann.ac.il/~dimagur/PSheaves2.html}