EXERCISE 3 IN PERVERSE SHEAVES

SHACHAR CARMELI

For an abelian category $\mathcal{C}$, let $K(\mathcal{C})$ denote the homotopy category of $\mathcal{C}$ and $D(\mathcal{C})$ the derived category. If $f : A \to B$ we denote by $\mathrm{Cone}(f)$ the mapping cone, and by $\pi_f : B \to \mathrm{Cone}(f)$ the inclusion to the second factor.

(1) Prove that $\mathrm{Cone}(\pi_f) \cong A[1]$.

(2) Prove that $H^i(A) \xrightarrow{f_*} H^i(B) \xrightarrow{(\pi_f)_*} H^i(\mathrm{Cone}(f))$ is exact. Deduce that a distinguished triangle induces a long exact sequence on cohomologies.

(3) (P) Prove that $f : A \to B$ is a homotopy equivalence if and only if $\mathrm{Cone}(f)$ is contractible (i.e. isomorphic to 0 in $K(\mathcal{C})$).

(4) Prove that $f : A \to B$ is a quasiisomorphism if and only if $\mathrm{Cone}(f)$ is acyclic.

(5) (*) Prove that every distinguished triangle in $K(\mathcal{C})$ is isomorphic to a short exact sequence. Deduce once again that a distinguished triangle induces a long exact sequence on cohomologies. Is every short exact sequence a distinguished triangle in $K(\mathcal{C})$? Prove that in $D(\mathcal{C})$, every short exact sequence is a distinguished triangle.

(6) (P) Prove that for $g : C \to B$ and $f : A \to B$, $g = f \circ g'$ if and only if $\pi_f \circ g = 0$. State and prove a dual statement.

(7) (P) Define the complex $Hom(A, B)$, and prove that $H^0(Hom(A, B)) = Hom_{K(\mathcal{C})}(A, B)$. Prove that $Hom(X, C(f)) = C(Hom(X, f))$. Use this to solve the last exercise much faster!

(8) (*) Prove that in $D^+(Ab)$ every object is isomorphic to a complex with zero differentials.

(9) (P) Prove that if $I \in K^+(\mathcal{C})$ is a complex of injective objects, and $M \in K^+(\mathcal{C})$ is acyclic, then $Hom_{K(C)}(M, I) = 0$. Deduce that for every complex A, $Hom_{D^+(\mathcal{C})}(A, I) \cong Hom_{K^+(\mathcal{C})}(A, I)$.

(10) Recall that $RHom(A, B) := Hom(A, I)$ where $B \cong I$ in $D^+(\mathcal{C})$ and I is a complex of injectives. Alternatively, if we work in $D^-(\mathcal{C})$ it is $Hom(P, A)$ where $A \cong P$ and P is a complex of projectives. Prove that $Hom_{D^+(\mathcal{C})}(A, B) = H^0(RHom(A, B))$, and similarly for $D^-(\mathcal{C})$.

(11) (*) Find an example of a complex which is not quasi-isomorphic to a complex with zero-differentials.

(12) (P) Let k be a field. Let $\mathcal{C} = Mod(k[x, y])$. Let A be the complex

$$k[x, y] \xrightarrow{\cdot x^2} k[x, y] \to k[x, y]/x$$

Date: May 17, 2022.

and B be the complex

$$k[x,y]/(y^2-x) \xrightarrow{\cdot y^2} k[x,y]/(y^2-x) \to k[x,y]/(x,y).$$

Compute the cohomologies of $RHom(A,B)$ and $A \overset{L}{\otimes} B$. Here, $RHom$ is the derived functor of the Hom functor, and $\overset{L}{\otimes}$ is the derived functor of the tensor product functor.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/PSheaves2.html