EXERCISE 2 IN INTRODUCTION TO REPRESENTATION THEORY

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(1) Let $\pi, \tau \in Rep(G)$ and let $\phi : \pi \to \tau$ be a morphism of representations which is an isomorphism of linear spaces. Show that ϕ is an isomorphism of representations. In other words, show that the linear inverse ϕ^{-1} is also a morphism of representations.
(2) (P) Show that a finite-dimensional representation π of a group G is a direct sum of irreducible representations if and only if for any subrepresentation $\tau \subset \pi$ there exists another subrepresentation $\tau' \subset \pi$ such that $\pi = \tau \oplus \tau'$.
(3) (P) Let G be an infinite group and $H < G$ a subgroup of finite index. Let (π, G, V) be a complex representation of G and $L \subset V$ a G-invariant subspace. Suppose we know that the subspace L has an H-invariant complement. Show that then it has a G-invariant complement.

Definition 1. *If X is a finite G-set we denote by π_X the natural representation of the group G on the space $F(X)$ of functions on X.*

(4) (P) Show that if X, Y are finite G-sets then the intertwining number $\langle \pi_X, \pi_Y \rangle$ equals to the number of G-orbits in the set $X \times Y$ (with respect to the diagonal action $g(x, y) = (gx, gy)$).
(5) Let $\pi \in Rep(G)$ and $\tau \in Rep(H)$. Let π^G denote the space of G-invariant vectors, $\pi^G = \{v \in \pi : \pi(g)v = v \, \forall g \in G\}$. Show that $(\pi \otimes \tau)^{G \times H} = \pi^G \otimes \tau^H$.
(6) Show that every complex matrix A with $A^n = Id$ is diagonalizable.
(7) (P) Let $\chi : G \to \mathbb{F}^\times = \mathrm{GL}_1(F)$ be a non-trivial one-dimensional representation of G. Show that $\sum_{g \in G} \chi(g) = 0$.
(8) (P) Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of basic quaternions. The product is given by $i^2 = j^2 = k^2 = ijk = -1$.
   (a) Show that $\pi : Q \to GL_2(\mathbb{C})$ defined by

$$\pi(\pm 1) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pi(\pm i) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pi(\pm j) = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pi(\pm k) = \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

   is an irreducible representation of Q.
   (b) Find four inequivalent one-dimensional representations of Q.
(9) (*) Let G, H be finite groups. Show that any irrep of $G \times H$ is of the form $\sigma \otimes \rho$, where $\sigma \in Irr(G)$, $\rho \in Irr(H)$.
(10) (*) We showed that $\langle \pi, \tau \rangle = \langle \tau, \pi \rangle$. Is that still true over
   (a) $F = \mathbb{R}$?
   (b) $F = \mathbb{F}_p$?

URL: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo5.html

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