# EXERCISE 4 IN INTRODUCTION TO REPRESENTATION THEORY

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**Fourier transform for finite groups.**

Let $C$ be a finite commutative group, $n = |C|$. We will denote by $\widehat{C}$ the dual group of characters $C \to \mathbb{C}$. We define Fourier transform $\mathcal{F} : F(C) \to F(\widehat{C})$ by $\mathcal{F}(u)(\psi) = \sum u(g)\psi(g)$.

(1) (P) Show that if we define an $L^2$-structure on spaces of functions by $||u||^2 = 1/n \sum |u(g)|^2$ then the operator $\mathcal{F}$ satisfies the Plancherel formula $||\mathcal{F}(u)||^2 = n||u||^2$.

(2) (P) Using the Plancherel formula prove the following Theorem(Gauss). Fix a non-trivial multiplicative character $\chi$ and a nontrivial additive character $\psi$ for the finite field $\mathbb{F}_q$ and consider the Gauss sum $\Gamma = \sum \chi(g)\psi(g)$, where the sum is taken over $g \in F^\times$ . Then $|\Gamma| = q^{1/2}$.

**Induction**

(3) (a) $H = \{e\}$, $\mathrm{Ind}_H^G(\mathbb{C}) = F(G)$.
  (b) For any $H, \mathrm{Ind}_H^G(\mathbb{C}) = F(G/H)$.
  (c) For any character $\chi$ of $H$, $\mathrm{Ind}_H^G(\chi) = \{f \in F(G) : f(gh) = \chi(h^{-1})f(g)$.

(4) (a) For $H < G$ and $\pi_1, \pi_2 \in Rep(H)$,
$$\mathrm{Ind}_H^G(\pi_1 \oplus \pi_2) = \mathrm{Ind}_H^G(\pi_1) \oplus \mathrm{Ind}_H^G(\pi_2).$$
  (b) For $H_1 < H_2 < G$ and $\pi \in Rep(H)$,
$$\mathrm{Ind}_{H_2}^G \mathrm{Ind}_{H_1}^{H_2} \pi = \mathrm{Ind}_{H_1}^G \pi$$

(5) (P) Let $G$ be a finite group, $D$ its subgroup and $\chi$ a character of $D$. Consider the induced representation $\pi = Ind_D^G(\chi)$. Show that $\pi$ is irreducible iff the following condition holds:
(*) For any $g \in G$ with $g \notin D$ there exists an element $x \in D$ such that the element $y = gxg^{-1}$ belongs to $D$ and $\chi(x) \neq \chi(y)$.

(6) (P) Barak has got an advanced game, where a usual game cube was replaced by an dodecahedron with numbers $1, ..., 12$ on its faces. Each time he lost, he replaced the number on each face by the average of its neighbors. What numbers will be written on the faces after 30 losses? What is the precision of your answer? The same exercise for an icosahedron.

*URL*: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo.html

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