EXERCISE 10 IN INTRODUCTION TO REPRESENTATION THEORY

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(1) (P) Let (π, B) be a continuous representation of a compact group K in a Banach space B, and let $\rho \in \mathrm{Irr}_f(K)$. In the lecture we defined the isotypic component $\pi_\rho := \rho \otimes \mathrm{Hom}_K(\rho, \pi)$. Note that π_ρ is a continuous representation of K and has a natural embedding into π.

Show that the image of this embedding coincides with the image of $\mathrm{End}_{\mathbb{C}}(\rho)$ in π under the composition of the matrix coefficient embedding $\mathrm{End}_{\mathbb{C}}(\rho) \hookrightarrow C(K)$ and the action map $C(K) \otimes \pi \to \pi$ given by $f \otimes v \to \pi(f)v$.

(2) Denote by P_n the space of all functions on the sphere S^2 that are restrictions of polynomials of degree n in $\mathbb{R}^3$. Show that $P_n \subset P_{n+2}$ and $\dim P_n = (n+1)(n+2)/2$.

(3) (P) Let $SO(2) \subset SO(3)$ denote the subgroup of rotations with respect to the z axis and identify $S^2 = SO(3)/SO(2)$. Show that $\dim(P_n)^{SO(2)} = [n/2] + 1$.
Hint: Show that $(P_n)^{SO(2)}$ is spanned by z^n, $z^{n-2}(x^2+y^2), ..., z^{n-2[n/2]}(x^2+y^2)^{[n/2]}$.

(4) (P) Let L_n denote the restriction to S^2 of the n-th Legandre polynomial:

$$L_n(z) = \frac{d^n}{dz^n}((z^2-1)^n).$$

Show that L_n is orthogonal to P_{n-2}.
Hint: Show that for any $SO(2)$-invariant function f on S we have

$$\int_S f(x)dx = \int_{-1}^{1} 2\pi z f(z) dz,$$

deduce that

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} 2\pi z f_1(z) \overline{f_2(z)} dz$$

and use integration by parts to show that $L_n(z)$ is orthogonal to all polynomials in z of degree smaller than n.

URL: http://www.wisdom.weizmann.ac.il/~dimagur/RepTheo4.html

Date: January 6, 2022.