EXERCISE 4 IN INTRODUCTION TO REPRESENTATION THEORY

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(1) For any representation π of a group G, the following are equivalent:
   (a) π is isotypic
   (b) All irreducible subrepresentations of π are isomorphic
   (c) If $\pi \simeq \omega \oplus \tau$ with $\langle \omega, \tau \rangle = 0$ then either $\omega = 0$ or $\tau = 0$.

(2) Let a (finite) group G act on a (finite) set X. Let π_X be the corresponding representation of G on $F(X)$, and let χ_X be the corresponding character. Show that for any $g \in G$, $\chi_X(g)$ equals the number of elements of X fixed by g.

(3) Show that the character of the regular representation equals $|G|$ at the identity of the group, and equals zero in all other points.

(4) (P) Compute the characters of all irreducible representations of S_4. Use the description we gave on lecture 3. Hint: $\chi_{\pi \oplus \tau} = \chi_\pi + \chi_\tau$.

   Classification of irreducible representations of S_n. Note that conjugate classes in S_n = partitions of n, i.e. sets $(\alpha_1, ..., \alpha_k)$ of natural numbers s.t. $\alpha_1 + ... + \alpha_k = n$ and $\alpha_1 \geq ... \geq \alpha_k$.

   Let X be a set of size n and $G = \mathrm{Sym}(X) = S_n$. Let us now find an irreducible representation for each partition $\alpha = (\alpha_1, ..., \alpha_k)$. Denote by X_α the set of all decompositions of the set X to subsets $X_1, .., X_k$ s.t. $|X_i| = \alpha_i$.

   Definition 1. $T_\alpha := F(X_\alpha), \quad T'_\alpha := sgn \cdot T_\alpha$.

   Definition 2. *Denote by* α^* *the partition given by* $\alpha^*_i := |\{j \, : \, \alpha_j \geq i\}$.

(5) Show that α^* is a partition and $(\alpha^*)^* = \alpha$. Show that the partial order defined in the lecture is indeed a (partial) order relation.

(6) (*) Show that
$$\langle T_\alpha, T'_\beta \rangle = \begin{cases} 0, & \alpha \not\leq \beta^*; \\ 1, & \alpha = \beta^*. \end{cases}$$

This implies that T_α and T'_α have a unique joint irreducible component U_α and that these components are different for different α. This gives a classification of all irreducible representations of S_n.

Fourier transform for finite groups.

Let C be a finite commutative group, $n = |C|$. We will denote by $\widehat{C}$ the dual group of characters $C \to \mathbb{C}$. We define Fourier transform $\mathcal{F} : F(C) \to F(\widehat{C})$ by
$$\mathcal{F}(u)(\psi) = \sum u(g)\psi(g).$$

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(P) Show that if we define an L^2-structure on spaces of functions by $||u||^2 = 1/n \sum |u(g)|^2$ then the operator $\mathcal{F}$ satisfies the Plancherel formula $||\mathcal{F}(u)||^2 = n||u||^2$.

(P) Using the Plancherel formula prove the following Theorem(Gauss). Fix a non-trivial multiplicative character χ and a nontrivial additive character ψ for the finite field $\mathbb{F}_q$ and consider the Gauss sum $\Gamma = \sum \chi(g)\psi(g)$, where the sum is taken over $g \in F^\times$. Then $|\Gamma| = q^{1/2}$.