EXERCISE 9 IN REPRESENTATION THEORY

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(1) (P) Let G be a finite group and p be a prime number. Show that for any $g \in G$ there exist unique $g_r, g_s \in G$ such that $g = g_r g_s = g_s g_r$, the order of g_r is prime to p and the order of g_s is a power of p.

(2) (P) Let Y be a finite set, $t \in \mathrm{Sym}(Y)$ be an element of order which is a power of p, and X be the set of fixed points of t. Then p divides $|Y| - |X|$.

Principal series representations of $\mathrm{SL}(2, \mathbb{F}_q)$

(3) (P) Let $G := \mathrm{SL}(2, \mathbb{F}_q)$ be the group of 2×2 matrices with entries in $\mathbb{F}_q$ and determinant 1. Let $X := \mathbb{F}_q^2 \setminus 0$. Consider the representation of G on $F(X)$, where F is some algebraically closed field of characteristic zero (e.g. $F = \mathbb{C}$). For any character χ of the multiplicative group $\mathbb{F}_q^\times$, let

$$V_\chi := \{f \in F(X) \, s.t. \, f(\lambda x) = \chi(\lambda) f(x)\}$$

Let Δ denote the standard anti-symmetric bilinear form on F^2. Define a linear operator $T : F(X) \to F(X)$ by

$$Tf(x) := \sum_{\{y \, s.t. \, \Delta(x,y)=1\}} f(y).$$

Show that

(a) V_χ is a subrepresentation of $F(X)$ and $F(X) = \bigoplus_\chi V_\chi$

(b) T is an intertwining operator (=morphism of representations) and T maps V_χ into $V_{\chi^{-1}}$.

(c) $\langle F(X), F(X) \rangle = 2q - 2$.

(d) V_χ is irreducible if $\chi \neq \chi^{-1}$ and is a sum of 2 non-isomorphic irreducible components if $\chi = \chi^{-1}$. Find the 2 components. Note that if q is odd there are only two χ with $\chi = \chi^{-1}$, and if q is even there is only one: $\chi = 1$.

(e) $V_\chi \simeq V_\psi$ if and only if $\chi = \psi^{\pm 1}$.

(f) T is invertible

(g) The sum of squares of the dimensions of the irreducible representations we found is $q(q-1)(q+3)/2$, which is approximately $|G|/2$.

The V_χ are called principal series representations and T is called the standard intertwining operator.

Date: January 22, 2022.