EFFECTIVE LOCAL DIFFERENTIAL TOPOLOGY OF ALGEBRAIC VARIETIES OVER LOCAL FIELDS OF POSITIVE CHARACTERISTICS

AVRAHAM AIZENBUD, DMITRY GOUREVITCH, DAVID KAZHDAN, AND EITAN SAYAG

ABSTRACT. In this paper we provide a framework for quantitative statements on distances and measures when studying algebraic varieties and morphisms of algebraic varieties over local fields.

We will concentrate on local fields of the type $\mathbb{F}_\ell((t))$ and work uniformly with respect to finite extensions of $\mathbb{F}_\ell$.

In this framework we prove analogues of standard results from local differential topology, including the implicit function theorem and study the behavior of smooth measures under push forward with respect to submersions.

CONTENTS

1. Introduction 2
1.1. The framework 2
1.2. Main results 2
1.3. Related results 5
1.4. Motivation 5
1.5. Ideas of the proofs 6
1.6. Structure of the paper 7
1.7. Acknowledgments 7
2. Conventions 7
3. Balls and measures on rectified varieties 8
4. Effective version of the implicit function theorem and its corollaries 13
5. m-smooth functions and measures 22
6. Effectively surjective maps 25
Index 29
References 29

Date: February 18, 2026.

2020 *Mathematics Subject Classification.* 54E35, 14G20, 26B10, 28C15.

Key words and phrases. implicit function theorem, smooth measures, algebraic varieties in positive characteristic.

1. Introduction

The goal of this paper is to provide a framework for formulating quantitative statements on distances and measures when studying algebraic varieties and maps between them over local fields.

We will concentrate on local fields of the type $\mathbb{F}_\ell((t))$ and work uniformly with respect to finite extensions of $\mathbb{F}_\ell$.

We introduce a notion of a rectification of an algebraic variety. This notion allows us to define the concept of a ball on a variety and to fix a family of measures on it.

1.1. The framework.

For a variety $\mathbf{X}$ defined over a finite field $\mathbb{F}_\ell$ we introduce the notion of rectification (see Definition 3.1). This notion allows us to define balls in the set $X := \mathbf{X}(F)$ of F-points of the variety, where F is a local field containing $\mathbb{F}_\ell$ (see Definition 3.3). Note that the notion of a ball is defined simultaneously for all local fields of the type $\mathbb{F}_{\ell^k}((t))$. This allows us to formulate uniform statements for all such fields.

Notation 1.1. *For a variety $\mathbf{X}$ and an integer $k \in \mathbb{N}$, we will consider two kinds of balls in $\mathbf{X}(\mathbb{F}_{\ell^k}((t)))$.*

1. *Non-centered balls, denoted by $B_m^{\mathbf{X},k}$, see Definition 3.3(1)(a) for the formal definition. These could be thought of as balls around the origin (though the origin is not necessarily a point in $\mathbf{X}$). Here $m \in \mathbb{Z}$ is the valuative radius of the ball, i.e. the actual radius is ℓ^{km}. Usually, m will be positive when considering such a ball.*
2. *Centered balls, denoted by $B_m^{\mathbf{X},k}(x)$, see Definition 3.3(1)(c) for the formal definition. These are balls of valuative radius $m \in \mathbb{Z}$ around $x \in \mathbf{X}(\mathbb{F}_{\ell^k}((t)))$. Here the integer m is usually negative.*

Although the balls themselves will depend on the rectification, all the statements that we will prove will not. This is due to the fact that for any two rectifications, one can compare between the corresponding balls. See Corollary 3.6(i).

Similarly, we will define the notion of a μ-rectification of smooth algebraic varieties (see Definition 3.1). This notion allows us to fix measures on balls in X (see Definition 3.3). Again, although the measures themselves will depend on the μ-rectification, the results that we will prove will not. This is established in Corollary 3.6(ii) and Lemma 5.5.

Remark 1.2. *For the sake of simplicity, we work only with algebraic varieties defined over $\mathbb{F}_\ell$. This is enough for our purposes. However, we believe that with minor modifications, all the statements would be valid also for varieties defined over $\mathbb{F}_\ell[[t]]$ and, with slightly more modifications, also for varieties defined over $\mathbb{F}_\ell((t))$.*

1.2. Main results.

We prove the following results:

1. Effective uniform continuity and boundedness of algebraic morphisms on balls. See §1.2.1.

(2) Effective version of the implicit function theorem. See §1.2.2.
(3) The compliment of a small neighborhood around a closed subvariety $\mathbf{Z} \subset \mathbf{X}$ is controlled by large balls in the complement of $\mathbf{Z}$. See §1.2.3.
(4) Effective surjectivity of Nisnevich covers. See §1.2.4.
(5) Effective smoothness of push forward of smooth measures with respect to smooth maps. See §1.2.5.
(6) Effective bounds on pushforward of smooth measures with respect to smooth maps. See §1.2.6.

1.2.1. *Effective uniform continuity and boundedness.* Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a map of algebraic varieties defined over a finite field $\mathbb{F}_\ell$. This gives maps $\gamma_k : \mathbf{X}(\mathbb{F}_{\ell^k}((t))) \to \mathbf{Y}(\mathbb{F}_{\ell^k}((t)))$. Note that each map γ_k is uniformly continuous and bounded on any ball in $\mathbf{X}(\mathbb{F}_{\ell^k}((t)))$. We prove that the modulus of continuity and the bound on γ_k in a ball $B_m^{\mathbf{X},k}$ of a fixed (valuative) radius m are bounded when we vary k.

More formally, we prove:

Proposition A (Proposition 3.5)**.** *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a map of rectified algebraic varieties defined over a finite field $\mathbb{F}_\ell$. Then for any $m \in \mathbb{N}$ there is $m' > m$ such that for any $k \in \mathbb{N}$ we have:*

(i) $\gamma(B_m^{\mathbf{X},k}) \subset B_{m'}^{\mathbf{X},k}$.
(ii) For any $x \in B_m^{\mathbf{X},k}$, we have $\gamma(B_{-m'}^{\mathbf{X},k}(x)) \subset B_{-m}^{\mathbf{X},k}(\gamma(x))$.

1.2.2. *Effective versions of the inverse and the implicit function theorems.* We prove an effective versions of the inverse and the implicit function theorems. Informally it means the following:

Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be an etale (respectively smooth) map of smooth algebraic varieties defined over a finite field $\mathbb{F}_\ell$. We again consider the maps $\gamma_k : \mathbf{X}(\mathbb{F}_{\ell^k}((t))) \to \mathbf{Y}(\mathbb{F}_{\ell^k}((t)))$. Then, in a ball $B_m^{\mathbf{X},k}$ the map γ_k admits a local inverse (respectively section), with bounded modulus of continuity, when $m \in \mathbb{N}$ is fixed and k varies.

More formally, we prove the following theorems.

Theorem B (Theorem 4.1)**.** *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be an étale map of smooth rectified algebraic varieties defined over a finite field $\mathbb{F}_\ell$. Then for any m there is m' such that $\gamma|_{B_m^{\mathbf{X},k}}$ is a monomorphism on balls of valuative radius $-m'$.*

Theorem C (Theorem 4.2)**.** *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a smooth map of smooth rectified algebraic varieties defined over a finite field $\mathbb{F}_\ell$. Then for any m there is m' such that for any k and any $x \in B_m^{\mathbf{X},k}$ we have*

$$\gamma(B_{-m}^{\mathbf{X},k}(x)) \supset B_{-m'}^{\mathbf{X},k}(\gamma(x)).$$

1.2.3. *Control on the compliment of a small neighborhood around a closed subvariety.*

Proposition D (Proposition 4.3)**.** *Let $\mathbf{X}$ be a rectified variety. Let $\mathbf{U} \subset \mathbf{X}$ be open and $\mathbf{Z} := \mathbf{X} \smallsetminus \mathbf{U}$. Then for any m there exists m' such that for any*

k, the ball $B^{\mathbf{X},k}_m$ is covered by the union of the ball $B^{\mathbf{U},k}_{m'}$ and the neighborhood of $\mathbf{Z}(\mathbb{F}_{\ell^k}((t)))$ of valuative radius $-m$.

Note that the notion of ball in $\mathbf{U}$ is not the one induced from $\mathbf{X}$, but rather an intrinsic notion for $\mathbf{U}$. In particular, $\mathbf{Z}$ is considered to be infinitely far with respect to this notion.

1.2.4. *Effectively surjective maps.* We introduce the notion of effective surjectivity for a map $\gamma : \mathbf{X} \to \mathbf{Y}$. Informally it means that the map is not only surjective on the level of points but we also can control the norm of a pre-image in terms of the norm of the target in a way that is uniform on extension of the local field. More precisely:

Definition 1.3 (Definition 6.1)**.** *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a map between rectified algebraic varieties defined over $\mathbb{F}_\ell$. We say that γ is effectively surjective if for any m there exist m' such that for any k we have*

$$\gamma(B^{\mathbf{X},k}_{m'}) \supset B^{\mathbf{Y},k}_m.$$

We prove the following criterion for this surjectivity:

Theorem E (Theorem 6.3)**.** *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a smooth map of rectified algebraic varieties defined over a finite field $\mathbb{F}_\ell$ that is onto on the level of points for any field extension of $\mathbb{F}_\ell$. Then γ is effectively surjective.*

Remark 1.4. *In fact our argument also shows that if γ is onto on the level of points for any infinite field extension of $\mathbb{F}_\ell$, then there is a finite extension of scalars of γ which is effectively surjective.*

1.2.5. *Effective smoothness of push forward of smooth measures with respect to smooth map.* Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a smooth map of smooth algebraic varieties defined over a finite field $\mathbb{F}_\ell$.

Let μ and ν be compactly supported measures on $X = \mathbf{X}(\mathbb{F}_\ell((t)))$ and $Y = \mathbf{Y}(\mathbb{F}_\ell((t)))$ which are coming from μ-rectifications on $\mathbf{X}$ and $\mathbf{Y}$. Assume that the support of ν includes the support of $\gamma_*(\mu)$. We show that the density of the pushforward $\gamma_*(\mu)$ with respect to ν is given by a smooth function which is constant on balls of some valuative radius $-m$. Moreover m remains bounded when we replace $\mathbb{F}_\ell$ with its finite extensions.

More formally, we make:

Notation 1.5 (Definition 3.3)**.** *For any integers m, k and a μ-rectified smooth algebraic variety defined over $\mathbb{F}_\ell$ we use the rectification in order to define a measure $\mu^{\mathbf{X},k}_m$ on $B^m_{\mathbf{X},k}$.*

We prove:

Theorem F (Theorem 5.7)**.** *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a smooth map of μ-rectified varieties. Then for any $m \in \mathbb{N}$ there is $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ and any function $g \in C^\infty_c(\mathbf{X}(\mathbb{F}_{\ell^k}((t))))$ which is constant on balls of valuative radius $-m$, there is a function $f \in C^\infty_c(\mathbf{Y}(\mathbb{F}_{\ell^k}((t))))$ which is constant on balls of valuative radius $-m'$, such that:*

$$\gamma_*(g\mu^{\mathbf{X},k}_m) = f \cdot \mu^{\mathbf{Y},k}_{m'}.$$

1.2.6. *Effective bounds on push forward of smooth measures.* We prove:

Theorem G (Lemma 3.9, Theorem 4.4, Corollary 6.8)**.** *Let* $\gamma : \mathbf{X} \to \mathbf{Y}$ *be a submersion of* μ*-rectified smooth varieties defined over* $\mathbb{F}_\ell$*. Then for any* m *there are* $m'' > m' > m$ *such that for any* $k \in \mathbb{N}$ *we have*

$$\ell^{-km'} \mu_m^{\mathbf{Y},k}|_{\mathrm{Supp}(\gamma_*(\mu_{m'}^{\mathbf{X},k}))} < \gamma_*(\mu_{m'}^{\mathbf{X},k}) < \ell^{km''} \mu_{m''}^{\mathbf{Y},k}$$

Moreover, if γ *is effectively surjective then*

$$\ell^{-km'} \mu_m^{\mathbf{Y},k} < \gamma_*(\mu_{m'}^{\mathbf{X},k})$$

1.3. **Related results.** Our notion of balls is parallel to the notion of norms in [Kot05, §18]. However, while [Kot05, §18] was concerned with large balls, we are also interested in small balls, and in measures. On the other hand, [Kot05, §18] needed more quantitative results than we need. It is likely that our theory can be put in a more general context.

The results of [Kot05] are, by themselves, not uniform on the field. Since [Kot05] allows non-local fields like $\bar{\mathbb{F}}_p((t))$, sometimes it is possible to deduce from it results that are uniform on field extensions. However, this is not the case for results that include the existential quantifier, like the implicit function theorem.

Other related notions of metric in the context of Archimedean algebraic geometry was introduced in [Wei21, BKS24] under the name of metric algebraic geometry.

1.4. **Motivation.** The main motivation for this work is a sequel work [AGKS] where we bound the dimensions of the jet schemes of the nilpotent cone (in $\mathfrak{gl}_n$) in positive (small) characteristic.

The characteristic zero counterpart of this result is done by [Mus01, Appendix by Eisenbud and Frenkel]. The methods of [Mus01] are not available in positive characteristic. Instead we use an analytic argument, resembling [HC70] in order to bound the number of points in these jet schemes. These arguments involve volumes and integration.

We use the Lang-Weil bounds to deduce the required bound on the dimension. For this we need the bound on the number of points to be uniform in field extensions of the underlying finite field. Classical analytic arguments do not give such uniform bounds. Therefore they are not enough for us and we need the results of the present paper.

In [AGKS], we use the bound on the dimension of the jet-schemes in order to bound the push-forward of smooth measures under the Chevalley map $p : \mathfrak{gl}_n \to \mathfrak{c}_n$, that sends every matrix to its characteristic polynomial. We do it under the assumption of existence of a certain resolution of singularities.

Note that the 0-characteristic counterpart of this result is done [HC70]. While we can use the method of [HC70] in a neighborhood of the nilpotent cone, the method fails to provide the desired global result due to issues of positive characteristic. This is why we take the detour through the jet-schemes.

1.5. **Ideas of the proofs.** A possible way to get uniform results for the fields $\mathbb{F}_{\ell^k}((t))$ is to work over the field $\bar{\mathbb{F}}_\ell((t))$. However, this field is not locally compact and we can not obtain boundedness in the standard way. Thus, we have to use different methods, as we will now describe.

The proof of Proposition A is straightforward but technical.

1.5.1. *Idea of the proofs of Theorem B and Theorem C.* The classical argument for the implicit function theorem (see e.g. [KP13]) is rather effective for the case of affine spaces, but not in general. So we have to employ additional considerations. Let us briefly explain the main steps in our proof:

Step 1. We prove Theorem B by reducing to the case of standard étale map.

Step 2. We prove Theorem C for the case of a map between (principal open subsets of) affine spaces.

Step 3. We deduce the case when the source $\mathbf{X}$ is a general variety and the target $\mathbf{Y}$ is an affine space. This we do by noticing that locally $\mathbf{X}$ can be written as a fiber of smooth map between affine spaces $\mathbf{V} \to \mathbf{W}$ and then applying the statement for the map $\mathbf{V} \to \mathbf{W} \times \mathbf{Y}$.

Step 4. The general case: we use the fact that locally $\mathbf{Y}$ can be mapped by an étale map to an affine space $\mathbf{L}$. Then we use the result for the composition $\mathbf{X} \to \mathbf{Y} \to \mathbf{L}$. In order to deduce the required statement, we also use Theorem B.

1.5.2. *Idea of the proof of Proposition D.* We use a stratification argument in order to reduce the statement to the case when $\mathbf{Z}$ is smooth. This case we deduce from Theorem C using the fact that $\mathbf{Z}$ is locally a fiber of a smooth map.

1.5.3. *Idea of the proof of Theorem E.*

Step 1. We show (using Noetherian induction) that if a map is surjective on the level of points over every field, then the target admits a stratification such that the map admits a section for each strata. See Lemma 6.4 below.

Step 2. We note that if a map admits a section then it is effectively surjective.

Step 3. We show, using the effective version of the implicit function theorem (Theorem C), that if a smooth map is effectively surjective over a closed subvariety, then it is effectively surjective over a controllable neighborhood of this subvariety.

Step 4. We use the previous step and Proposition D to show that any smooth map that is effectively surjective over a closed subvariety and over its complement is effectively surjective.

Step 5. We deduce the theorem.

1.5.4. *Idea of the proof of Theorem F.* Here we use a standard technique of presentation of a smooth map as a composition of simpler maps, and the effective version of the implicit function theorem, Theorem C.

1.5.5. *Idea of the proof of Theorem G.* The main part is the first inequality. Here the main difficulty is that we can not deduce the statement for a composition of morphisms from the statement for each morphism. However, we still can reduce to the case when the morphism is a composition of an étale map and a projection from a product with affine space. For such compositions we can explicitly compute the density function of the push forward. Then we can get the required bound using Theorem C.

1.6. **Structure of the paper.** In §2 we fix the conventions that we use in this paper.

In §3 we define the notion of rectified algebraic varieties, the notion of a ball in such varieties and define the measures $\mu_m^{\mathbf{X},k}$ described above. We also show that these objects essentially do not depend on the rectification. We also prove basic properties of these objects, including Proposition A, and the second (easier) inequality of Theorem G.

In §4 we prove effective version of the implicit function theorem (Theorem C) and draw some corollary of it (Proposition D and the main part of Theorem G).

In §5 we prove Theorem F.

In §6 we prove Theorem E and complete the proof of Theorem G.

1.7. **Acknowledgments.** We would like to thank Gal Binyamini and Yosef Yomdin for helpful discussions.

During the preparation of this paper, A.A., D.G. and E.S. were partially supported by the ISF grant no. 1781/23. D.K. was partially supported by the ERC grant no. 101142781.

2. Conventions

(1) Throughout we fix a prime power ℓ.
(2) By a variety we mean a reduced scheme of finite type over a field. Unless stated otherwise this field will be $\mathbb{F}_\ell$.
(3) Morphisms between varieties will be always defined over the field of definition of the varieties.
(4) When we consider a fiber product of varieties, and fibers of maps between varieties, we always consider it in the category of schemes.
(5) We will describe subschemes and morphisms of varieties and schemes using set-theoretical language, when no ambiguity is possible.
(6) We will usually denote algebraic varieties by bold face letters (such as $\mathbf{X}$).
(7) We will use the symbol $\square$ in a middle of a square diagram in order to indicate that the square is Cartesian.
(8) For a field extension E/F and a variety $\mathbf{X}$ defined over F, we will denote by $\mathbf{X}_E$ the extension of scalars. We will use analogous notation for all algebro-geometric objects on the variety $\mathbf{X}$, including regular functions and differential forms.

(9) For a regular function f on an algebraic variety $\mathbf{X}$ defined over a local field F, we denote by $|f|$ the corresponding function on $X := \mathbf{X}(F)$.
(10) For a top form ω on a smooth algebraic variety $\mathbf{X}$ defined over a local field F, we denote the corresponding measure on $X := \mathbf{X}(F)$ by $|\omega|$.

3. Balls and measures on rectified varieties

In this section we introduce the concept of rectified varieties and define balls on them. We also fix a measure on each ball.

We prove that the balls and measures are essentially independent of the rectification, see Corollary 3.6.

We also prove some basic properties of these objects, including Proposition A and the second (easier) inequality of Theorem G.

Definition 3.1. *Let $\mathbf{X}$ be a smooth algebraic variety over $\mathbb{F}_\ell$.*

(1) A rectification of $\mathbf{X}$ is a finite open cover $\mathbf{X} \subset \bigcup_{\alpha \in I} \mathbf{U}_\alpha$ with closed embeddings $i_\alpha : \mathbf{U}_\alpha \to \mathbb{A}^M$ for each $\alpha \in I$.
(2) We will call a rectification simple if $|I| = 1$.
(3) By a rectified variety we will mean a smooth algebraic variety over $\mathbb{F}_\ell$ equipped with a rectification. By a map or a morphism of such we just mean a morphism of the underlying algebraic varieties.
(4) A μ-rectification of $\mathbf{X}$ is a rectification of $\mathbf{X}$ together with invertible top differential forms $\omega_\alpha \in \Omega^{top}(\mathbf{U}_\alpha)$ for each $\alpha \in I$.
(5) We define similarly the notion of a μ-rectified variety, and simple μ-rectification.

Definition 3.2. *By an almost affine space we mean a principal open subset[1] in an affine space defined over $\mathbb{F}_\ell$. Note that any almost affine space is equipped with a natural simple (μ-)rectification that we will call the standard (μ-)rectification on this space.*

We are now ready to define balls and measures on our rectified varieties.

Definition 3.3.

(1) Let $(\mathbf{X}, \mathbf{U}_\alpha, i_\alpha)$ be a rectified variety. Then, for any $k \in \mathbb{N}$ and $m \in \mathbb{Z}$ define:
  (a) $B_m^{\mathbf{X},k} := \bigcup_\alpha i_\alpha^{-1}\left(t^{-m}\mathbb{F}_{\ell^k}[[t]]^M\right) \subset \mathbf{X}(\mathbb{F}_{\ell^k}((t)))$.
  (b) $B_\infty^{\mathbf{X},k} := \bigcup_{m \in \mathbb{N}} B_m^{\mathbf{X},k} = \mathbf{X}(\mathbb{F}_{\ell^k}((t)))$.
  (c) For $x \in \mathbf{X}(\mathbb{F}_{\ell^k}((t)))$ define a ball around x:

$$B_m^{\mathbf{X},k}(x) := \bigcup_{\alpha \ s.t. \ x \in U_\alpha(F_{\ell^k}((t)))} i_\alpha^{-1}\left(i_\alpha(x) + t^{-m}\mathbb{F}_{\ell^k}[[t]]^M\right).$$

  (d) For $\mathbf{Z} \subset \mathbf{X}$ we define:

$$B_m^{\mathbf{X},k}(\mathbf{Z}) := \bigcup_{z \in B_\infty^{\mathbf{Z},k}} B_m^{\mathbf{X},k}(z).$$

[1] i.e. the complement of a divisor

(2) Let $(\mathbf{X}, \mathbf{U}_\alpha, i_\alpha, \omega_\alpha)$ be a μ-rectified variety. Then, for any integers $k \in \mathbb{N}, m \in \mathbb{Z}$ we define a measure $\mu_m^{\mathbf{X},k}$ on $\mathbf{X}(\mathbb{F}_{\ell^k}((t)))$ supported on $B_m^{\mathbf{X},k}$ by:

$$\mu_m^{\mathbf{X},k} := \sum_\alpha |(\omega_\alpha)_{\mathbb{F}_{\ell^k}((t))}| \cdot 1_{i_\alpha^{-1}\left(t^{-m}\mathbb{F}_{\ell^k}[[t]]^M\right)}.$$

(3) If $\mathbf{X}$ is an affine space, we denote by $\mu^{\mathbf{X},k}$ the Haar measure on $\mathbf{X}(\mathbb{F}_{\ell^k}((t)))$ normalized such that $\mu^{\mathbf{X},k}(\mathbf{X}(\mathbb{F}_{\ell^k}[[t]]) = 1$.

The following is obvious:

Lemma 3.4. *Let $\mathbf{X}$ be a rectified variety. Then for any 2 positive integers $m_1, m_2 \in \mathbb{N}$ we have:*

(1) If $x \in B_{m_2}^{\mathbf{X},k}$ then $B_{-m_1}^{\mathbf{X},k}(x) \subset B_{m_2}^{\mathbf{X},k}$.
(2) If $x \in B_\infty^{\mathbf{X},k}$ and $y \in B_{-m_1}^{\mathbf{X},k}(x)$ then $x \in B_{-m_1}^{\mathbf{X},k}(y)$.
(3) If the rectification of $\mathbf{X}$ is simple and $m_1 \geq m_2$, then

$$B_{-m_1}^{\mathbf{X},k}(x) \subset B_{-m_2}^{\mathbf{X},k}(x) = B_{-m_2}^{\mathbf{X},k}(y)$$

for any $x \in \mathbf{X}(\mathbb{F}_{\ell^k}((t)))$ and $y \in B_{-m_2}^{\mathbf{X},k}(x)$.

Proposition 3.5. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be map of rectified algebraic varieties. Then for any $m \in \mathbb{N}$ there is $m' > m$ such that for any k and any $x \in B_m^{\mathbf{X},k}$ we have*

(i) $\gamma(B_m^{\mathbf{X},k}) \subset B_{m'}^{\mathbf{Y},k}$.
(ii) $\gamma(B_{-m'}^{\mathbf{X},k}(x)) \subset B_{-m}^{\mathbf{Y},k}(\gamma(x))$.

Proof.

Case 1. $\mathbf{X} = \mathbb{A}^M, \mathbf{Y} = \mathbb{A}^1$ (both equipped with the standard rectifications) and γ is a monomial:
In this case it is easy to see that one can take $m' = dm$ where d is the degree of γ.

Case 2. $\mathbf{X} = \mathbb{A}^M, \mathbf{Y} = \mathbb{A}^1$ (both equipped with the standard rectifications): follows from the previous case and the fact that $\mathbb{F}_{\ell^k}((t))$ is non Archimedean.

Case 3. $\mathbf{X}, \mathbf{Y}$ are affine spaces (both equipped with the standard rectifications):
follows from the previous case.

Case 4. The rectifications of $\mathbf{X}, \mathbf{Y}$ are simple:
by definition of a map between affine varieties we have a commutative diagram

$$\begin{array}{ccc} \mathbf{X} & \longrightarrow & \mathbb{A}^{M_1} \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma'} \\ \mathbf{Y} & \longrightarrow & \mathbb{A}^{M_2} \end{array}$$

where the rectifications of $\mathbf{X}$ and $\mathbf{Y}$ are induced from their embedding into affine spaces in the diagram. Now, this case follows from the previous one.

Case 5. The rectification of $\mathbf{Y}$ is simple:
Follows from the previous case.

Case 6. The rectification of $\mathbf{X}$ is simple, γ is an isomorphism, and the rectification of $\mathbf{Y}$ comes from a cover with principal open sets, and their standard embedding into $\mathbf{X} \times \mathbb{A}^1$:
Let $\mathbf{X} = \mathbf{Y} = \bigcup_{i=1}^{M} \mathbf{Y}_{f_i} = \bigcup \mathbf{X}_{f_i}$ be the cover defining the rectification of $\mathbf{Y}$. By Hilbert's Nullstellensatz we can find $g_i \in O_{\mathbf{X}}(\mathbf{X})$ such that $\sum f_i g_i = 1$. Consider the map $\mathbf{X} \to \mathbb{A}^{2M}$ given by $x \mapsto (f_1(x), \ldots, f_M(x), g_1(x), \ldots, g_M(x))$ and apply to it Case 4. We obtain numbers $m_1 > m_0 > m$ such that for any $x \in B_m^{\mathbf{X},k}$ we have
(a) $|g_i(x)| < |t^{-m_0}|$
(b) $f_i(B_{-m_0}^{\mathbf{X},k}(x)) \subset B_{-m-2m_0}^{\mathbb{A}^1,k}(f_i(x))$
Take $m' = m_1$. From (a) we obtain that for any $x \in B_m^{\mathbf{X},k}$ there exists i such that

$$|f_i(x)| > |t^{m_0}|. \tag{3.1}$$

Thus

$$x \in B_{m_0}^{\mathbf{Y}_{f_i},k} \subset B_{m_0}^{\mathbf{Y},k} \subset B_{m'}^{\mathbf{Y},k}$$

proving (i).

From (3.1) and (b) we obtain that for any $x \in B_m^{\mathbf{X},k}$ there exists i such that for any $y \in B_{-m'}^{\mathbf{X},k}(x)$ we have

$$\left| \frac{1}{f_i(x)} - \frac{1}{f_i(y)} \right| = \left| \frac{f_i(y) - f_i(x)}{f_i(x) f_i(y)} \right| < |f_i(y) - f_i(x)| \cdot |t^{-2m_0}| \leq |t^m|.$$

Thus

$$y \in B_{-m}^{\mathbf{Y}_{f_i},k}(x) \subset B_{-m}^{\mathbf{Y},k}(x),$$

proving (ii).

Case 7. The rectification of $\mathbf{X}$ is obtained from a union of rectifications of each $\gamma^{-1}(\mathbf{U}_i)$ where $\mathbf{Y} = \bigcup \mathbf{U}_i$ given by the rectification of $\mathbf{Y}$:
Follows immediately from Case 4.

Case 8. The rectification of $\mathbf{X}$ is simple, γ is an isomorphism.
Let $\mathbf{Y}_1$ be identical to $\mathbf{Y}$ as a variety but with a rectification defined by a cover by principal open sets such that the identity map $\mathbf{Y}_1 \to \mathbf{Y}$ satisfies the condition of the previous case (Case 7). The statement follows now from the previous 2 cases.

Case 9. γ is an isomorphism and the rectification of $\mathbf{Y}$ is obtained from a union of rectifications of each $\gamma(\mathbf{U}_i)$ where $\mathbf{X} = \bigcup \mathbf{U}_i$ given by the rectification of $\mathbf{X}$:
Follows from the previous case (Case 8).

Case 10. γ is an isomorphism.
Follows from Cases 7 and 9.

Case 11. general case:
Follows from Cases 7 and 10.

□

Corollary 3.6. *Let $\mathbf{X}_1, \mathbf{X}_2$ be two copies of the same $\mathbb{F}_\ell$-variety with two (possibly different) rectifications. Let $\mathbf{Z} \subset \mathbf{X}_1$ be a closed subvariety. Then*

(i) for any $m \in \mathbb{N}$ there is $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ we have:
  (a) $B_m^{\mathbf{X}_1,k} \subset B_{m'}^{\mathbf{X}_2,k}$
  (b) for any $x \in \mathbf{X}_1(\mathbb{F}_{\ell^k}((t)))$ we have $B_m^{\mathbf{X}_1,k}(x) \subset B_{m'}^{\mathbf{X}_2,k}(x)$
  (c) $B_m^{\mathbf{X}_1,k}(\mathbf{Z}) \subset B_{m'}^{\mathbf{X}_2,k}(\mathbf{Z})$.

(ii) For any μ-rectifications of $\mathbf{X}_i$ and $m \in \mathbb{N}$, there exists m' such that for any k we have:

$$\mu_m^{\mathbf{X}_1,k} < \ell^{km'} \mu_{m'}^{\mathbf{X}_2,k}.$$

Proof. Items (i) follow immediately from the previous proposition (Proposition 3.5). It is enough to prove (ii). We will proceed by analyzing cases:

Case 1. The μ-rectifications of $\mathbf{X}_i$ are simple, and their embedding into an affine space is the same:
Let ω_i be the form on $\mathbf{X}_i$. Let $g = \frac{\omega_1}{\omega_2} \in O^\times(\mathbf{X}_1)$. Fix m. By Proposition 3.5(i) there is $m' > m$ such that for any $k \in \mathbb{N}$ we have

$$\max(\mathrm{val}(g(B_m^{\mathbf{X}_1,k}))) < m'.$$

Now we have:

$$\mu_m^{\mathbf{X}_1,k} = \mu_m^{\mathbf{X}_2,k}|g| < \ell^{km'} \mu_m^{\mathbf{X}_2,k} \leq \ell^{km'} \mu_{m'}^{\mathbf{X}_2,k}.$$

Case 2. The μ-rectifications of $\mathbf{X}_i$ are simple and the forms on $\mathbf{X}_i$ are the same:
follows immediately from Proposition 3.5(i).

Case 3. The μ-rectifications of $\mathbf{X}_i$ are simple:
Let $\mathbf{X}_3$ be a variety identical to $\mathbf{X}_1, \mathbf{X}_2$ with a rectification given by the embedding of $\mathbf{X}_1$ to an affine space and the form on $\mathbf{X}_2$. The assertion follows now from the previous cases (applied to the pairs $(\mathbf{X}_1, \mathbf{X}_3)$ and $(\mathbf{X}_3, \mathbf{X}_2)$ in correspondence).

Case 4. The covers of $\mathbf{X}_1$ and $\mathbf{X}_2$ are the same:
Follows from the previous case.

Case 5. There is an invertible top form ω on $\mathbf{X}_1$ such that the forms in the rectifications on both $\mathbf{X}_1$ and $\mathbf{X}_2$ are restrictions of ω.
Let M be the size of largest of the 2 covers of $\mathbf{X}_i$. We have

$$|\omega_{\mathbb{F}_{\ell^k}((t))}| 1_{B_m^{\mathbf{X}_i,k}} \leq \mu_m^{\mathbf{X}_i,k} \leq M |\omega_{\mathbb{F}_{\ell^k}((t))}| 1_{B_m^{\mathbf{X}_i,k}}.$$

The assertion follows now from part (a).

Case 6. $\mathbf{X}_1$ admits an invertible top-form:
Follows from the previous 2 cases.

Case 7. The rectification of $\mathbf{X}_1$ is obtained by rectifications of each $\mathbf{U}_i$ where $\mathbf{X}_2 = \bigcup \mathbf{U}_i$ is the cover of $\mathbf{X}_2$ given by its rectification:
Follows from the previous case.

Case 8. The rectification of $\mathbf{X}_2$ is obtained by a rectification of each $\mathbf{U}_i$ where $\mathbf{X}_1 = \bigcup \mathbf{U}_i$ is the cover of $\mathbf{X}_1$ given by its rectification:
Follows from Case 6.

Case 9. General case:
Follows from the last 2 cases.

□

At some point we will need the following stronger version of Corollary 3.6 (a):

Lemma 3.7. *Let $\mathbf{X}_1, \mathbf{X}_2$ be two copies of the same $\mathbb{F}_\ell$-variety with two (possibly different) rectifications. Then there exists $a \in \mathbb{N}$ such that for any $m, k \in \mathbb{N}$ we have:*

$$B_m^{\mathbf{X}_1,k} \subset B_{am+a}^{\mathbf{X}_2,k}$$

Proof. In fact, the proof of Proposition 3.5 provides a proof of this lemma. It also follows from [Kot05, Proposition 18.1 (1)]. □

Corollary 3.8. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a closed embedding. Then for any integer $m \in \mathbb{N}$ there exists $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ we have $\gamma^{-1}(B_m^{\mathbf{Y},k}) \subset B_{m'}^{\mathbf{X},k}$*

Proof. Let $\mathbf{X}'$ be the variety $\mathbf{X}$ with the induced rectification from $\mathbf{Y}$. We have $\gamma^{-1}(B_m^{\mathbf{Y},k}) = B_m^{\mathbf{X}',k}$. The assertion follows now from Corollary 3.6(a). □

Lemma 3.9. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a submersion of μ-rectified varieties. Then for any m there is m' such that*

$$\gamma_*(\mu_m^{\mathbf{X},k}) < \ell^{km'} \mu_{m'}^{\mathbf{Y},k}.$$

Proof.

Case 1. The rectifications of $\mathbf{X}$ and of $\mathbf{Y}$ are simple, γ is étale and the form on $\mathbf{X}$ is the pullback of the form on $\mathbf{Y}$:
Fix $m \in \mathbb{N}$. By [Sta25, Tag 03JA, Tag 03J5] there is $M \in \mathbb{N}$ such that for any $y \in \mathbf{Y}(\bar{\mathbb{F}}_\ell((t)))$ we have

$$M > \#\gamma^{-1}(y).$$

By Proposition 3.5 there exists $m_1 > m$ such that

$$\gamma(B_m^{\mathbf{X},k}) \subset B_{m_1}^{\mathbf{Y},k}.$$

Take $m' = m_1 M$. Let $k \in \mathbb{N}$. Let $\omega_{\mathbf{Y}}$ be the form on $\mathbf{Y}$ given by its μ-rectification. For $k \in \mathbb{N}$, let $\omega_{\mathbf{Y},k} := (\omega_{\mathbf{Y},k})_{\mathbb{F}_\ell((t))}$ We have

$$\gamma_*(\mu_m^{\mathbf{X},k}) = f \cdot |\omega_{\mathbf{Y},k}|$$

where

$$f(y) = \#\{x \in B_m^{\mathbf{X},k} | \gamma(x) = y\}.$$

So,

$$\gamma_*(\mu_m^{\mathbf{X},k}) = f \cdot |\omega_{\mathbf{Y},k}| \le M \cdot 1_{\gamma(B_m^{\mathbf{X},k})} \cdot |\omega_{\mathbf{Y},k}| \le M \cdot 1_{B_{m_1}^{\mathbf{Y},k}} \cdot |\omega_{\mathbf{Y},k}| = M \cdot \mu_{m_1}^{\mathbf{Y},k} < \ell^{m'} \cdot \mu_{km'}^{\mathbf{Y},k}$$

Case 2. the rectifications of $\mathbf{X}$ and of $\mathbf{Y}$ are simple and γ is étale:
Follows from the previous case using Corollary 3.6.

Case 3. the μ-rectifications of $\mathbf{X}$ and of $\mathbf{Y}$ are simple, $\mathbf{X} = \mathbf{Y} \times \mathbb{A}^M$, and γ is the projection:
WLOG assume that the μ-rectification of $\mathbf{X}$ is given by the μ-rectification of $\mathbf{Y}$ and the standard μ-rectification of $\mathbb{A}^M$. Take $m' = mM + 1$. The assertion is a straightforward computation.

Case 4. $\gamma = \gamma_1 \circ \gamma_2$ when γ_1 satisfy the conditions of Case 3 and γ_2 satisfy the conditions of Case 2:
Follows immediately from the previous 2 cases.

Case 5. The rectification of $\mathbf{Y}$ is simple:
By [Sta25, Tag 039P] there is a cover $\mathbf{X} = \bigcup_{i=1}^M \mathbf{U}_i$ such that $\gamma|_{\mathbf{U}_i}$ satisfy the condition of the previous case. WLOG we can also assume that each $\mathbf{U}_i$ admits a simple μ-rectification. By Corollary 3.6, we can also assume that the μ-rectification of $\mathbf{X}$ is coming from these rectifications. The statement follows now from the previous case.

Case 6. General case:
Follows from the previous case and Corollary 3.6.

∎

Corollary 3.10. *Let $\mathbf{X}$ be a μ-rectified variety. Then for any $m \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that for any $k \in \mathbb{N}$:*

$$\mu_m^{\mathbf{X},k}(B_\infty^{\mathbf{X},k}) < \ell^{kM}.$$

Proof. It follows from the previous lemma for the map $\gamma : \mathbf{X} \to pt = \mathbb{A}^0$. ∎

4. Effective version of the implicit function theorem and its corollaries

In this subsection we prove Theorem B (see Theorem 4.1 below) and Theorem C (see Theorem 4.2 below).

We also deduce 2 statements:

- Proposition D. See Proposition 4.3.
- The main part of Theorem G: The push of the measure $\mu_m^{\mathbf{X},k}$ under a submersion $\mathbf{X} \to \mathbf{Y}$ controls (from above) the measure $\mu_m^{\mathbf{Y},k}$ on the support of the former. See Theorem 4.4.

We start with the following:

Theorem 4.1. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be an étale map of smooth (rectified) algebraic varieties defined over $\mathbb{F}_\ell$. Then for any m there is m' such that for any k and any $x \in B_m^{\mathbf{X},k}$ the map $\gamma|_{B_{-m'}^{\mathbf{X},k}(x)}$ is a monomorphism.*

Proof. By Corollary 3.6 the statement does not depend on the rectification, so we will choose it each time as convenient. For an integer n, denote by $\mathbf{Y}_n$ the collection of monic polynomials of degree n, considered as an affine space. Denote also $\mathbf{X}_n := \{(f, a) | f \in \mathbf{Y}_n; f(a) = 0, f'(a) \neq 0\}$.

Case 1. $\mathbf{X} = \mathbf{X}_n$, $\mathbf{Y} = \mathbf{Y}_n$ (for some n), and γ is the projection:
Embed $\mathbf{X}$ into $\mathbb{A}^{n+2}$ by $(f, a) \mapsto (f, a, \frac{1}{f'(a)})$. This gives a rectification of $\mathbf{X}$. Take also the rectification of $\mathbf{Y}$ that comes from the fact that it is an affine space.

Set $m' = mn + 1 + m + 1$. Let $x = (f, a) \in B_m^{\mathbf{X},k}$. It is enough to show that if $(f, b) \in B_{-m'}^{\mathbf{X},k}(x)$ then $a = b$. For a polynomial $f \in \mathbb{Z}[x]$ denote $f^{[k]} := \frac{f^{(k)}}{k!}$. This is a polynomial over $\mathbb{Z}$, so this operation is defined over any field. We have

$$0 = f(b) = f(a + (b - a)) = f(a) + f^{[1]}(a)(b - a) + \cdots = f^{[1]}(a)(b - a) + \cdots$$

Assuming that $b \neq a$ we obtain:

$$f^{[1]}(a) + (b - a)f^{[2]}(a) + \cdots = 0$$

Since $x \in B_m^{\mathbf{X},k}$, we have

$$|f^{[i]}(a)| \leq |t^{-mn}|.$$

Thus

$$|(b - a)f^{[2]}(a) + \ldots| < |t|^{m+1}.$$

On the other hand

$$f^{[1]}(a) = f'(a) \geq |t|^m.$$

This leads to a contradiction.

Case 2. γ is standard étale map[2]:
Follows from the previous step, Proposition 3.5 and the fact that we have a base change diagram:

$$\begin{array}{ccc} \mathbf{X} & \longrightarrow & \mathbf{X}_n \\ {\scriptstyle\gamma}\big\downarrow & \square & \big\downarrow \\ \mathbf{Y} & \longrightarrow & \mathbf{Y}_n \end{array}$$

Case 3. General case:
Follows from the previous case using [Sta25, Tag 02GT].

□

The next result is an effective version of the open mapping theorem:

Theorem 4.2. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a smooth map of smooth (rectified) algebraic varieties. Then for any $m \in \mathbb{N}$ there is $m' > m$ such that for any k and any $x \in B_m^{\mathbf{X},k}$ we have*

$$\gamma(B_{-m}^{\mathbf{X},k}(x)) \supset B_{-m'}^{\mathbf{Y},k}(\gamma(x)).$$

Proof.

[2]See [Sta25, Definition 02GI].

Case 1. $\mathbf{X} = (\mathbb{A}^d)_f$ is an almost affine space, $\mathbf{Y} = \mathbb{A}^d$:

Let $j \in O(\mathbf{X})$ be the Jacobian of γ. Choose the rectification of $\mathbf{X}$ that is given by the embedding $x \mapsto (x, f(x)^{-1}, j(x)^{-1})$, and choose the standard rectificaion of $\mathbf{Y}$. Fix m. Write

$$\gamma(x) = \frac{h(x)}{f(x)^M} = \frac{\sum_{|i|\leq M} a_i x^i}{f(x)^M},$$

in multi-index notation, where a_i are vectors. Note that $\mathrm{val}(a_i) = 0$, where the valuation of a vector is defined to be the minimum of the valuations of its coordinates. Let

$$m' = mM + m(\deg(f)M + M + \deg(j)) + 8mdM\deg(f) + Mm.$$

Take $y' \in B^{\mathbf{Y},k}_{-m'}(\gamma(x))$. We have to find $x' \in B^{\mathbf{X},k}_{-m}(x)$ such that $\gamma(x') = y'$. Define recursively a sequence $x_r \in B^{\mathbf{X},k}_{-m}(x)$. Set $x_0 = x$, and define

$$x_{r+1} := x_r + (D_{x_r}\gamma)^{-1}(y' - \gamma(x_r)).$$

We will show by induction on r that for any $r \geq 0$:

(a_r) $\mathrm{val}(y' - \gamma(x_r)) \geq m' + r$

(b_r) $x_r \in B^{\mathbf{X},k}_{m}$

(c_r) $x_{r+1} \in B^{\mathbf{X},k}_{-m-r}(x_r)$

Before proving these statements, we will show that for any given r statements (a_r) and (b_r) imply (c_r). For this note that for $w \in \mathbb{F}_{l^k}((t))^d$ we have

$$\begin{aligned}
\mathrm{val}\left((D_{x_r}\gamma)^{-1}w\right) &= \mathrm{val}\left(\det\left(D_{x_r}\gamma\right)^{-1} Adj\left(D_{x_r}\gamma\right)w\right) = \mathrm{val}\left(j(x_r)^{-1} Adj\left(D_{x_r}\gamma\right)w\right) \\
&\geq \mathrm{val}\left(j(x_r)^{-1}\right) + \mathrm{val}(w) + (d-1)\min_{1\leq s,t\leq d}\mathrm{val}\left((D_{x_r}\gamma)_{s,t}\right) \\
&= \mathrm{val}\left(j(x_r)^{-1}\right) + \mathrm{val}(w) + (d-1)\min_{1\leq s,t\leq d}\mathrm{val}\left(\partial_t\gamma_s(x_r)\right) \\
&= \mathrm{val}\left(j(x_r)^{-1}\right) + \mathrm{val}(w) + (d-1)\min_{1\leq s,t\leq d}\mathrm{val}\left(\partial_t\left(\tfrac{h_s}{f^M}\right)(x_r)\right) \\
&= \mathrm{val}\left(j(x_r)^{-1}\right) + \mathrm{val}(w) + (d-1)\min_{1\leq s,t\leq d}\mathrm{val}\left(\tfrac{f^M(x_r)\partial_t h_s(x_r) - \partial_t(f^M)(x_r)h_s(x_r)}{f(x_r)^{2M}}\right) \\
&= \mathrm{val}\left(j(x_r)^{-1}\right) + \mathrm{val}(w) + \\
&+ (d-1)\min_{1\leq s,t\leq d}\left(\mathrm{val}\left(f(x_r)^M\partial_t h_s(x_r) - \partial_t(f^M)(x_r)h_s(x_r)\right) - \mathrm{val}\left(f(x_r)^{2M}\right)\right) \\
&= \mathrm{val}\left(j(x_r)^{-1}\right) + \mathrm{val}(w) + \\
&+ (d-1)\min_{1\leq s,t\leq d}\left(\min(\mathrm{val}\left(f(x_r)^M\partial_t h_s(x_r)\right), \mathrm{val}\left(\partial_t(f^M)(x_r)h_s(x_r)\right)\right) + \\
&+ \min\left(-\mathrm{val}\left(f(x_r)^{2M}\right)\right)
\end{aligned}$$

Statement (b_r) implies that:

$$\mathrm{val}\left(j(x_r)^{-1}\right) + \mathrm{val}(w) + (d-1)\min_{1\leq s,t\leq d}\left(\min(\mathrm{val}\left(f(x_r)^M\partial_t h_s(x_r)\right), \mathrm{val}\left(\partial_t(f^M)(x_r)h_s(x_r)\right)\right) +$$

$$+ \min\left(-\mathrm{val}\left(f(x_r)^{2M}\right)\right) \geq -m - d(\deg(f)M + M)m - 2dMm \geq -4mdM\deg(f)$$

So

$$\text{val}((D_{x_r}\gamma)^{-1}w) \geq -4mdM\deg(f) + \text{val}(w). \tag{4.1}$$

This implies:

$$\text{val}(x_{r+1} - x_r) = \text{val}((D_{x_r}\gamma)^{-1}(y' - \gamma(x_r))) \geq -4mdM\deg(f) + \text{val}(y' - \gamma(x_r))$$

So, by (a_r) we obtain:

$$\text{val}(x_{r+1} - x_r) \geq -4mdM\deg(f) + m' + r \geq 3m + m\deg(f) + m\deg(j) + r \tag{4.2}$$

This implies that

$$\text{val}(f(x_{r+1}) - f(x_r)) \geq \text{val}(x_{r+1} - x_r) - m\deg(f) \geq 3m + r. \tag{4.3}$$

So

$$\text{val}(f(x_{r+1})) = \text{val}((f(x_{r+1}) - f(x_r)) + f(x_r)) \leq m.$$

Thus

$$\text{val}\left(\frac{1}{f(x_{r+1})} - \frac{1}{f(x_r)}\right) = \text{val}\left(\frac{f(x_r) - f(x_{r+1})}{f(x_{r+1})f(x_r)}\right) > m + r. \tag{4.4}$$

Similarly,

$$\text{val}\left(\frac{1}{j(x_{r+1})} - \frac{1}{j(x_r)}\right) > m + r. \tag{4.5}$$

Formulas (4.2),(4.4) and (4.5) (which are proven under the assumptions of (a_r) and (b_r)) imply (c_r).

We now proceed to prove by induction (a_r), (b_r) and (c_r). The base statements (a_0), and (b_0) are obvious. So the base statement (c_0) follows from the above. For the induction step we assume (a_r), (b_r) and (c_r) for an integer r and prove (a_{r+1}), (b_{r+1}) and (c_{r+1}).

By Lemma 3.4(1) statement (b_{r+1}) follows from (c_r) and (b_r). Also, as shown before, (a_{r+1}) and (b_{r+1}) imply (c_{r+1}). It is left to show (a_{r+1}) and we can use (b_{r+1}) for this. For $z \in \mathbb{F}_{l^k}((t))^d$, write

$$\gamma(x_r + z) = \gamma(x_r) + (D_{x_r}\gamma)z + \frac{\sum_{2\leq|i|\leq M} b_i z^i}{f(x_r + z)^M}.$$

Let $\delta(w) := f(w)^M(\gamma(w) - \gamma(x_r) - (D_{x_r}\gamma)(w - x_r))$. Note that

- this is a polynomial map.
- The valuation of its coefficients are bounded from below by $-mM$.
- Its degree is bounded by $\deg(f)M + M$.

Define $\delta^{[i]}$ as in the proof of Theorem 4.1, but for vector valued functions of a vector variable. We have

$$\text{val}(b_i) = \text{val}(\delta^{[i]}(x_r)) \geq -mM - m(\deg(f)M + M).$$

Setting

$$z := (D_{x_r}\gamma)^{-1}(y' - \gamma(x_r)),$$

formula (4.1) above implies

$$\operatorname{val}(z) \geq -4mdM\deg(f) + \operatorname{val}(y' - \gamma(x_r)).$$

So,

$$\begin{aligned}
\operatorname{val}(y' - \gamma(x_{r+1})) &= \operatorname{val}(y' - \gamma(x_r + z)) = \\
&= \operatorname{val}\left(y' - \left(\gamma(x_r) + (D_{x_r}\gamma)z + \frac{\sum_{2\leq|i|\leq M} b_i z^i}{f(x_r+z)^M}\right)\right) = \\
&= \operatorname{val}\left(y' - \gamma(x_r) - (D_{x_r}\gamma)z - \frac{\sum_{2\leq|i|\leq M} b_i z^i}{f(x_r+z)^M}\right) = \\
&= \operatorname{val}\left(\frac{\sum_{2\leq|i|\leq M} b_i z^i}{f(x_r+z)^M}\right) = \operatorname{val}\left(\sum_{2\leq|i|\leq M} b_i z^i\right) - \operatorname{val}\left(f(x_r+z)^M\right) \\
&= \operatorname{val}\left(\sum_{2\leq|i|\leq M} b_i z^i\right) - \operatorname{val}\left(f(x_{r+1})^M\right) \\
&\geq \min_{2\leq|i|\leq M}(\operatorname{val}(b_i) + |i|\operatorname{val}(z)) - \operatorname{val}\left(f(x_{r+1})^M\right) \\
&\geq -mM - m(\deg(f)M + M) + 2\operatorname{val}(z) - M\operatorname{val}(f(x_{r+1})) \\
&\geq -mM - m(\deg(f)M + M) - 8mdM\deg(f) + 2\operatorname{val}(y' - \gamma(x_r)) - \\
&\quad - M\operatorname{val}(f(x_{r+1})).
\end{aligned}$$

By (b_{r+1}), we know that

$$\operatorname{val}(f(x_{r+1})^{-1}) \geq -m.$$

By (a_r), we know that

$$\operatorname{val}(y' - \gamma(x_r)) \geq m' + r.$$

Thus

$$\begin{aligned}
\operatorname{val}(y' - \gamma(x_{r+1})) &\geq -mM - m(\deg(f)M + M) - 8mdM\deg(f) + 2m' + 2r - Mm \geq \\
&\geq m' + r + 1,
\end{aligned}$$

proving (a_{r+1}) and finishing the proof of (a_r), (b_r) and (c_r) for every r.

Now, (c_r) implies that $\operatorname{val}(x_{r+1} - x_r) \geq m + r$. Thus the sequence x_r converges. Let $x' := \lim_{r\to\infty} x_r$. Using Lemma 3.4(3) and (c_r) again, we get that $x_r \in B^{\mathbf{X},k}_{-m}(x)$ for any r. Thus $x' \in B^{\mathbf{X},k}_{-m}(x)$. Finally, $(a_r)_{r\in\mathbb{N}}$ implies that $\gamma(x') = y'$ as required.

Case 2. $\mathbf{X}$ and $\mathbf{Y}$ are almost affine spaces and $\dim(\mathbf{X}) = \dim(\mathbf{Y})$:
Follows immediately from the previous case.

Case 3. $\mathbf{X}$ and $\mathbf{Y}$ are almost affine spaces, and there exists a morphism $p : \mathbf{X} \to \mathbb{A}^{\dim\mathbf{X}-\dim\mathbf{Y}}$ such that the map $\gamma \times p : \mathbf{X} \to \mathbf{Y} \times \mathbb{A}^{\dim\mathbf{X}-\dim\mathbf{Y}}$

is étale:
Follows from the previous case applied to the map $\gamma \times p$.

Case 4. $\mathbf{X}$ is an open subset of an affine space and $\mathbf{Y}$ is almost affine spaces:
We can find a finite cover $\mathbf{X} := \bigcup \mathbf{U}_i$ such that $\gamma|_{\mathbf{U}_i}$ satisfies the conditions of the previous case. The statement now follows from the previous case.

Case 5. $\mathbf{Y}$ is an almost affine space and $\mathbf{X}$ is a fiber of a smooth map between almost affine spaces:
Write $\mathbf{X}$ as the fiber of a smooth map $\mathbf{U} \to \mathbf{V}$. Extend the map γ to a map $\gamma' : \mathbf{U} \to \mathbf{Y}$, this defines a map $\gamma'' : \mathbf{U} \to \mathbf{V} \times \mathbf{Y}$. Note that γ'' is smooth at $\mathbf{X}$. Thus we can find $\mathbf{U}' \subset \mathbf{U}$ containing $\mathbf{X}$ such that $\gamma''|_{\mathbf{U}'}$ is smooth. The statement follows now from the previous case applied to $\gamma''|_{\mathbf{U}'}$.

Case 6. $\mathbf{Y}$ is an almost affine space:
We can find a finite cover $\mathbf{X} := \bigcup \mathbf{U}_i$ such that $\gamma|_{\mathbf{U}_i}$ satisfies the conditions of the previous case. The statement now follows from the previous case.

Case 7. $\mathbf{Y}$ is an affine variety that admits an étale map to an affine space:
Let $\delta : \mathbf{Y} \to \mathbf{U}$ be an étale map to an almost affine space. Fix m. By Proposition 3.5 there is $m_1 > m$ such that for any $k \in \mathbb{N}$ we have

$$\gamma(B_m^{\mathbf{X},k})) \subset B_{m_1}^{\mathbf{Y},k}. \tag{4.6}$$

By Theorem 4.1 there is $m_2 > m_1$ such that for any $k \in \mathbb{N}$ and any $x \in B_{m_1}^{\mathbf{Y},k}$ we have

$$\delta|_{B_{-m_2}^{\mathbf{Y},k}(x)} \quad \text{is a monomorphism.} \tag{4.7}$$

By Proposition 3.5 there is $m_3 > m_2$ such that for any $k \in \mathbb{N}$ and any $x \in B_m^{\mathbf{X},k}$ we have

$$\gamma(B_{-m_3}^{\mathbf{X},k}(x)) \subset B_{-m_2}^{\mathbf{Y},k}(\gamma(x)). \tag{4.8}$$

By the previous case there is $m_4 > m_3$ such that for any $k \in \mathbb{N}$ and any $x \in B_m^{\mathbf{X},k}$ we have

$$\delta(\gamma(B_{-m_3}^{\mathbf{X},k}(x))) \supset B_{-m_4}^{\mathbf{U},k}(\delta(\gamma(x))). \tag{4.9}$$

By Proposition 3.5 there is $m_5 > m_4$ such that for any $k \in \mathbb{N}$ and any $x \in B_{m_5}^{\mathbf{Y},k}$ we have

$$\delta(B_{-m_5}^{\mathbf{Y},k}(x)) \subset B_{-m_4}^{\mathbf{U},k}(\delta(x)). \tag{4.10}$$

Take $m' = m_5$. Let k be an integer and $x \in B_m^{\mathbf{X},k}$. Let $y' \in B_{-m'}^{\mathbf{Y},k}(\gamma(x))$. we have to find $x' \in B_{-m}^{\mathbf{X},k}(x)$ such that $\gamma(x') = y'$.

By (4.6) $\gamma(x) \in B_{m_1}^{\mathbf{Y},k}$. So, by (4.10)

$$\delta(y') \in B_{-m_4}^{\mathbf{U},k}(\delta(\gamma(x))).$$

Thus, by (4.9), there is $x' \in B_{-m_3}^{\mathbf{X},k}(x)$ such that

$$\delta(\gamma(x')) = \delta(y'). \tag{4.11}$$

By (4.8)

$$\gamma(x') \in B^{\mathbf{Y},k}_{-m_2}(\gamma(x)).$$

Also

$$y' \in B^{\mathbf{Y},k}_{-m_2}(\gamma(x)).$$

Therefore, (4.7) and (4.11) imply that $\gamma(x') = y'$ as required.

Case 8. General case:
We find a finite cover $\mathbf{Y} = \bigcup \mathbf{U}_i$ such that the maps $\gamma_i : \gamma^{-1}(\mathbf{U}_i) \to \mathbf{U}_i$ obtained by the restriction of γ to $\mathbf{U}_i$ satisfy the conditions of the previous case. So the previous case implies the assertion.

□

Proposition 4.3. *Let* $\mathbf{X}$ *be a rectified variety. Let* $\mathbf{U} \subset \mathbf{X}$ *be open and* $\mathbf{Z} := \mathbf{X} \smallsetminus \mathbf{U}$*. Then for any* $m \in \mathbb{N}$ *there exists* $m' > m$ *such that for any* k *we have*

$$B^{\mathbf{X},k}_{m} \subset B^{\mathbf{U},k}_{m'} \cup B^{\mathbf{X},k}_{-m}(\mathbf{Z}).$$

Proof.

Case 1. the rectification of $\mathbf{X}$ is simple, $\mathbf{Z}$ is a fiber of a smooth map $\delta : \mathbf{X} \to \mathbf{Y}$, and $\mathbf{Y}$ has simple rectification:
Let $x_1, \ldots, x_d$ be the coordinates of the affine space that includes $\mathbf{Y}$. WLOG assume that $\mathbf{Z}$ is the fiber of $0 \in \mathbf{Y}(\mathbb{F}_\ell)$. Let $f_i := \delta^*(x_i)$. Embed $\mathbf{X}_i := \mathbf{X}_{f_i}$ to affine space in the standard way, and choose the rectification of $\mathbf{U}$ given by these embeddings. It is easy to see that for any integers m', k we have

$$B^{\mathbf{U},k}_{m'} = B^{\mathbf{X},k}_{m'} \smallsetminus \delta^{-1}(B^{\mathbf{Y},k}_{-m'-1}(0)).$$

Indeed,

$$\begin{aligned} B^{\mathbf{U},k}_{m'} &= \bigcup_i B^{\mathbf{X}_i,k}_{m'} = \bigcup_i \{x \in \mathbf{X}_i(\mathbb{F}_{\ell^k}((t))) | x \in B^{\mathbf{X},k}_{m'}; \mathrm{val}(f_i(x)^{-1}) \geq -m'\} = \\ &= \bigcup_i (B^{\mathbf{X},k}_{m'} \smallsetminus \{x \in \mathbf{X}(\mathbb{F}_{\ell^k}((t))) | \mathrm{val}(f_i(x)) > m'\}) = \\ &= B^{\mathbf{X},k}_{m'} \smallsetminus \bigcap_i \{x \in \mathbf{X}(\mathbb{F}_{\ell^k}((t))) | \mathrm{val}(f_i(x)) > m'\} = \\ &= B^{\mathbf{X},k}_{m'} \smallsetminus \delta^{-1}(B^{\mathbf{Y},k}_{-m'-1}(0)). \end{aligned}$$

Fix m. By Theorem 4.2 there exists $m_1 > m$ such that for any $k \in \mathbb{N}$ and any $x \in B^{\mathbf{X},k}_{m}$ we have

$$\delta(B^{\mathbf{X},k}_{-m}(x)) \supset B^{\mathbf{Y},k}_{-m_1}(\delta(x)).$$

Take $m' = m_1 + 1$. It is left show that

$$\delta^{-1}(B^{\mathbf{Y},k}_{-m'-1}(0)) = \delta^{-1}(B^{\mathbf{Y},k}_{-m_1}(0)) \subset B^{\mathbf{X},k}_{-m}(\mathbf{Z}).$$

Let $x \in \delta^{-1}(B^{\mathbf{Y},k}_{-m_1}(0))$. We have to show that $x \in B^{\mathbf{X},k}_{-m}(\mathbf{Z})$. We have

$$0 \in B^{\mathbf{Y},k}_{-m_1}(\delta(x)) \subset \delta(B^{\mathbf{X},k}_{-m}(x)).$$

So we have $z \in B^{\mathbf{X},k}_{-m}(x)$ such that $\delta(z) = 0$. By Lemma 3.4 this implies that $x \in B^{\mathbf{X},k}_{-m}(z)$. Also $z \in \mathbf{Z}(F_{\ell^k}((t)))$. So $x \in B^{\mathbf{X},k}_{-m}(B^{\mathbf{X},k}_{m} \cap \mathbf{Z})$, as required.

Case 2. $\mathbf{Z}$ is smooth.

Follows from the previous step and from the fact that the question is local on $\mathbf{X}$.

Case 3. General case

We will prove the statement by induction on $\dim \mathbf{Z}$. Let $\mathbf{Z}'$ be the singular locus of $\mathbf{Z}$. Let $\mathbf{U}' = \mathbf{X} \setminus \mathbf{Z}'$. Choose rectification of $\mathbf{U}'$ such that each ball in $\mathbf{U}'$ is contained in the corresponding ball in $\mathbf{X}$. Fix m. By the induction assumption there exist $m_1 > m$ such that for any k we have

$$B^{\mathbf{X},k}_{m} \subset B^{\mathbf{U}',k}_{m_1} \cup B^{\mathbf{X},k}_{-m}(\mathbf{Z}')$$

By the previous case there exist $m_2 > m_1$ such that for any k we have

$$B^{\mathbf{U}',k}_{m_1} \subset B^{\mathbf{U},k}_{m_2} \cup B^{\mathbf{U}',k}_{-m_1}((\mathbf{U}' \cap \mathbf{Z}))$$

Take $m' = m_2$. We get

$$\begin{aligned} B^{\mathbf{X},k}_{m} &\subset B^{\mathbf{U}',k}_{m_1} \cup B^{\mathbf{X},k}_{-m}(\mathbf{Z}') \subset \\ &\subset B^{\mathbf{U},k}_{m_2} \cup B^{\mathbf{U}',k}_{-m_1}((\mathbf{U}' \cap \mathbf{Z})) \cup B^{\mathbf{X},k}_{-m}(\mathbf{Z}') \\ &\subset B^{\mathbf{U},k}_{m_2} \cup B^{\mathbf{X},k}_{-m}((\mathbf{U}' \cap \mathbf{Z})) \cup B^{\mathbf{X},k}_{-m}(\mathbf{Z}') \\ &\subset B^{\mathbf{U},k}_{m_2} \cup B^{\mathbf{X},k}_{-m}(\mathbf{Z}) \end{aligned}$$

as required.

□

Theorem 4.4. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a submersion of μ-rectified varieties. Then for any $m, m' \in \mathbb{N}$ there is $M \in \mathbb{N}$ such that for any k we have*

$$\mu^{\mathbf{Y},k}_{m} \cdot 1_{\mathrm{Supp}(\gamma_*(\mu^{\mathbf{X},k}_{m'}))} < \ell^{kM} \gamma_*(\mu^{\mathbf{X},k}_{m'})$$

Proof.

Case 1. the μ-rectifications of $\mathbf{X}$ and of $\mathbf{Y}$ are simple, γ is étale and the form on $\mathbf{X}$ is the pullback of the form on $\mathbf{Y}$:

Take $M = 1$. Let $\omega_{\mathbf{Y}}$ be the form on $\mathbf{Y}$ given by its μ-rectification. For $k \in \mathbb{N}$, let $\omega_{\mathbf{Y},k} := (\omega_{\mathbf{Y}})_{\mathbb{F}_{\ell^k}((t))}$ be the corresponding form on the extension of scalars $\mathbf{Y}_{\mathbb{F}_{\ell^k}((t))}$. We have

$$\gamma_*(\mu^{\mathbf{X},k}_{m'}) = f \cdot |\omega_{\mathbf{Y}}|,$$

where

$$f(y) = \#\{x \in B^{\mathbf{X},k}_{m'} | \gamma(x) = y\}.$$

Now, we have:

$$\begin{aligned} \mu^{\mathbf{Y},k}_{m} \cdot 1_{\mathrm{Supp}(\gamma_*(\mu^{\mathbf{X},k}_{m'}))} &= 1_{B^{\mathbf{Y},k}_{m}} \cdot 1_{\mathrm{Supp}(\gamma_*(\mu^{\mathbf{X},k}_{m'}))} \cdot |\omega_{\mathbf{Y},k}| = \\ &= 1_{B^{\mathbf{Y},k}_{m}} \cdot 1_{\mathrm{Supp}(f)} \cdot |\omega_{\mathbf{Y},k}| < \ell^k \cdot f \cdot |\omega_{\mathbf{Y},k}| = \ell^{kM} \gamma_*(\mu^{\mathbf{X},k}_{m'}), \end{aligned}$$

as required.

Case 2. the μ-rectifications of $\mathbf{X}$ and of $\mathbf{Y}$ are simple and γ is étale:
Follows from the previous case and Corollary 3.6.

Case 3. the rectifications of $\mathbf{X}$ and of $\mathbf{Y}$ are simple and $\gamma = \gamma_1 \circ \gamma_2$ where $\gamma_1 : \mathbf{Y} \times \mathbb{A}^j \to \mathbf{Y}$ is a projection and γ_2 is étale:
Choose the μ-rectification of $\mathbf{Y} \times \mathbb{A}^j$ that is coming from the μ-rectification of $\mathbf{Y}$ and the standard rectification of $\mathbb{A}^j$. Let $\omega_{\mathbf{Y},k}$ be as above.

Fix $m, m' \in \mathbb{N}$. By Proposition 3.5 (applied to the map γ_2) there exists m_1 such that

$$\gamma_2(B^{\mathbf{X},k}_{m'}) \subset B^{\mathbf{Y}\times\mathbb{A}^j,k}_{m_1}.$$

By the previous case we have $M_1 \in \mathbb{N}$ such that for any k we have

$$\mu^{\mathbf{Y}\times\mathbb{A}^j,k}_{m_1} \cdot 1_{\mathrm{Supp}((\gamma_2)_*(\mu^{\mathbf{X},k}_{m'}))} < \ell^{kM_1}(\gamma_2)_*(\mu^{\mathbf{X},k}_{m'}).$$

By Theorem 4.2 (applied to the map γ_2) we have m_2 such that for any $z \in \gamma_2(B^{\mathbf{X},k}_{m'})$ we have

$$B^{\mathbf{Y}\times\mathbb{A}^j,k}_{-m_2}(z) \subset \gamma_2(B^{\mathbf{X},k}_{m'}). \tag{4.12}$$

Take $M = M_1 + Nm_2$. We have

$$(\gamma_1)_*(\mu^{\mathbf{Y}\times\mathbb{A}^j,k}_{m_1} \cdot 1_{\gamma_2(B^{\mathbf{X},k}_{m'})}) = g \cdot |\omega_{\mathbf{Y},k}|$$

where

$$\begin{aligned} g(y) :&= Vol(\{v \in \mathbb{F}_{\ell^k}((t))^j | (y,v) \in B^{\mathbf{Y}\times\mathbb{A}^j,k}_{m_1} \cap \gamma_2(B^{\mathbf{X},k}_{m'})\}) = \\ &= Vol(\{v \in \mathbb{F}_{\ell^k}((t))^j | (y,v) \in \gamma_2(B^{\mathbf{X},k}_{m'})\}). \end{aligned}$$

Note that by (4.12) if $g(y) \neq 0$ then

$$g(y) \geq Vol(B^{\mathbb{A}^n,k}_{-m_2}(0)) = \ell^{-kNm_2}.$$

Thus

$$\begin{aligned} \mu^{\mathbf{Y},k}_m \cdot 1_{\mathrm{Supp}(\gamma_*(\mu^{\mathbf{X},k}_{m'}))} &= \mu^{\mathbf{Y},k}_m \cdot 1_{\mathrm{Supp}(g)} \leq \mu^{\mathbf{Y},k}_m \cdot g\ell^{kNm_2} \leq |\omega_{\mathbf{Y},k}| \cdot g\ell^{kNm_2} = \\ &= (\gamma_1)_*(\mu^{\mathbf{Y}\times\mathbb{A}^j,k}_{m_1} \cdot 1_{\gamma_2(B^{\mathbf{X},k}_{m'})})\ell^{kNm_2} = \\ &= (\gamma_1)_*(\mu^{\mathbf{Y}\times\mathbb{A}^j,k}_{m_1} \cdot 1_{\mathrm{Supp}((\gamma_2)_*(\mu^{\mathbf{X},k}_{m'}))})\ell^{kNm_2} < \\ &< (\gamma_1)_*(\ell^{kM_1}(\gamma_2)_*(\mu^{\mathbf{X},k}_{m'}))\ell^{kNm_2} = \\ &= \ell^{kM_1+kNm_2}\gamma_*(\mu^{\mathbf{X},k}_{m'}) = \ell^{kM}\gamma_*(\mu^{\mathbf{X},k}_{m'}) \end{aligned}$$

Case 4. The rectification of $\mathbf{Y}$ is simple:
By [Sta25, Tag 039P] there is a cover $\mathbf{X} = \bigcup_{i=1}^{j} \mathbf{U}_i$ such that $\gamma|_{\mathbf{U}_i}$ satisfy the condition of the previous case. The statement follows now from the previous case and Corollary 3.6.

Case 5. General case:
follows from the previous case and Corollary 3.6.

□

5. m-SMOOTH FUNCTIONS AND MEASURES

In this subsection we give a quantitative notion for smoothness of functions and measures, and prove that this notion behaves well under push. In particular we prove Theorem F.

Definition 5.1. *Let* $\mathbf{X}$ *be a rectified variety. Let* $m, k \in \mathbb{N}$. *We say that* $f \in C_c^\infty(B_\infty^{\mathbf{X},k})$ *is* m*-smooth if for any* $x \in B_\infty^{\mathbf{X},k}$ *the function* $f|_{B_{-m}^{\mathbf{X},k}(x)}$ *is constant.*

Lemma 5.2 (Criteria for m-smoothness)**.** *Let* $\mathbf{X}$ *be a rectified variety. Let* $m, k \in \mathbb{N}$.

(1) Let $f \in C_c^\infty(B_\infty^{\mathbf{X},k})$ *be a real valued function such that for any* $x \in B_\infty^{\mathbf{X},k}$ *we have*

$$\min(f(B_{-m}^{\mathbf{X},k}(x))) = f(x).$$

Then f *is* m*-smooth.*

(2) Let $f \in C_c^\infty(B_\infty^{\mathbf{X},k})$ *be a function such that for any* $x \in \operatorname{Supp}(f)$ *we have* $f|_{B_{-m}^{\mathbf{X},k}(x))}$ *is constant. Then* f *is* m*-smooth.*

Proof. Follows immediately from Lemma 3.4(2). □

Lemma 5.3. *Let* $\mathbf{X}$ *be a rectified algebraic variety and* $\mathbf{U} \subset \mathbf{X}$ *be an open subset. Fix a rectification on* $\mathbf{U}$. *Let* $f \in C^\infty(B_m^{\mathbf{U},k})$ *be an* m*-smooth function. Then there is* m' *such that* $f \in C^\infty(B_{m'}^{\mathbf{X},k})$ *is an* m'*-smooth function.*

Proof. Fix $m \in \mathbb{N}$. By Proposition 3.5(i) there exists $m_1 > m$ such that for any $k \in \mathbb{N}$ we have $B_m^{\mathbf{U},k} \subset B_{m_1}^{\mathbf{X},k}$. By Theorem 4.2 there exists $m' > m_1$ such that for any $k \in \mathbb{N}$ and any $x \in \operatorname{Supp}(f) \subset B_m^{\mathbf{U},k}$ we have $B_{-m}^{\mathbf{U},k}(x) \supset B_{-m'}^{\mathbf{X},k}(x)$. By Lemma 5.2(2) this implies the assertion. □

Lemma 5.4. *Let* $\mathbf{X}$ *be a rectified algebraic variety and* $f \in \mathcal{O}^\times(\mathbf{X})$. *Then for any* $m \in \mathbb{N}$ *there exists* $m' > m$ *such that for any* $k \in \mathbb{N}$ *the function* $|f| 1_{B_m^{\mathbf{X},k}}$ *on* $B_\infty^{\mathbf{X},k}$ *is* m'*-smooth.*

Proof. Consider f as a function to $\mathbb{A}^1 \setminus 0$. By Proposition 3.5(i) there is $m_1 > m$ such that for any $k \in \mathbb{N}$ we have

$$\max(\operatorname{val}(f(B_m^{\mathbf{X},k}))) < m_1.$$

By Proposition 3.5(ii) we obtain that there is $m_2 > m_1$ such that for any $k \in \mathbb{N}$ and any $x \in B_m^{\mathbf{X},k}$ we have

$$\min(\operatorname{val}(f(B_{-m_2}^{\mathbf{X},k}(x)) - f(x))) > m_1.$$

Take $m' = m_2$. We obtain that for any $k \in \mathbb{N}$ and any $x \in B_m^{\mathbf{X},k}$ the function $\operatorname{val}(f)|_{B_{-m'}^{\mathbf{X},k}}$ is constant. This implies the assertion. □

Lemma 5.5. *Let* $\gamma : \mathbf{X}_1 \to \mathbf{X}_2$ *be an open embedding of* μ*-rectified varieties. Then for any* $m \in \mathbb{N}$ *there is* $m' \in \mathbb{N}$ *such that for any* $k \in \mathbb{N}$ *there is a* m'*-smooth function* $f_k \in C_c^\infty(B_{m'}^{\mathbf{X}_2,k})$ *such that:*

$$\gamma_*(\mu_m^{\mathbf{X}_1,k}) = f_k \cdot \mu_{m'}^{\mathbf{X}_2,k}$$

Proof.

Case 1. The μ-rectification of $\mathbf{X}_1$ is simple: Let $\mathbf{X}_2 = \bigcup \mathbf{U}_i$ be the cover of $\mathbf{X}_2$. Let ω_i be the form on $\mathbf{U}_i$ and ν be the form on $\mathbf{X}_1$. Let $g_i = \frac{\gamma^*(\omega_i)}{\nu} \in O^\times(\mathbf{X}_1)$. Fix $m \in \mathbb{N}$. By Proposition 3.5(i) there is $m_1 > m$ such that for any $k \in \mathbb{N}$ we have $\gamma(B_m^{\mathbf{X}_1,k}) \subset \gamma(B_{m_1}^{\mathbf{X}_2,k})$. For every $k \in \mathbb{N}$ and every i, denote

$$f_{i,k} := |(g_i)_{\mathbb{F}_{\ell^k}((t))}| \cdot 1_{B_m^{\mathbf{X}_1,k} \cap \gamma^{-1}(B_{m_1}^{\mathbf{U}_i,k})} \in C_c^\infty(B_\infty^{\mathbf{X}_1,k}).$$

Then we have

$$(\mu_{m_1}^{\mathbf{U}_i,k})1_{B_m^{\mathbf{X}_1,k}} = f_{i,k}\mu_m^{\mathbf{X}_1,k}.$$

From the previous lemma (Lemma 5.4), there is $m_2 > m_1$ such that for any $k \in \mathbb{N}$ the function $f_{i,k}$ is m_2-smooth.

Let $h_k = \sum_i f_{i,k}$. We obtain:

- $\gamma^*(\mu_{m_1}^{\mathbf{X}_2,k})1_{B_m^{\mathbf{X}_1,k}} = h_k\mu_m^{\mathbf{X}_1,k}$.
- h_k is m_2-smooth.

Thus,

$$(\mu_{m_1}^{\mathbf{X}_2,k})1_{\gamma(B_m^{\mathbf{X}_1,k})} = (\mu_{m_1}^{\mathbf{X}_2,k})\gamma_*(1_{B_m^{\mathbf{X}_1,k}}) = \gamma_*(\gamma^*(\mu_{m_1}^{\mathbf{X}_2,k})1_{B_m^{\mathbf{X}_1,k}}) = \gamma_*(h_k\mu_m^{\mathbf{X}_1,k}) = \gamma_*(h_k)\gamma_*(\mu_m^{\mathbf{X}_1,k}).$$

Note that

$$\mathrm{Supp}(h_k) = \bigcup \mathrm{Supp}(f_{i,k}) = \bigcup (B_m^{\mathbf{X}_1,k} \cap \gamma^{-1}(B_{m_1}^{\mathbf{U}_i,k})) = B_m^{\mathbf{X}_1,k} \cap \bigcup \gamma^{-1}(B_{m_1}^{\mathbf{U}_i,k}) =$$
$$= B_m^{\mathbf{X}_1,k} \cap \gamma^{-1}(B_{m_1}^{\mathbf{X}_2,k}) = B_m^{\mathbf{X}_1,k}$$

Let

$$f_k(x) = \begin{cases} \gamma_*(h_k)^{-1}(x), & \text{if } x \in \mathrm{Supp}(\gamma_*(h_k)) \\ 0, & \text{otherwise} \end{cases}$$

We get

$$(\mu_{m_1}^{\mathbf{X}_2,k})f_k = (\mu_{m_1}^{\mathbf{X}_2,k})1_{\gamma(B_m^{\mathbf{X}_1,k})}f_k = f_k\gamma_*(h_k)\gamma_*(\mu_m^{\mathbf{X}_1,k}) = 1_{\gamma(B_m^{\mathbf{X}_1,k})}\gamma_*(\mu_m^{\mathbf{X}_1,k}) = \gamma_*(\mu_m^{\mathbf{X}_1,k}).$$

So it remains to show that there is m' such that for any k the function f_k is an m'-smooth function on $B_\infty^{\mathbf{X}_2,k}$. This follows from the above using Lemma 5.3.

Case 2. the general case

Follows from the previous case.

□

Lemma 5.6. *Let $\gamma : \mathbf{X}_1 \to \mathbf{X}_2$ be an étale morphism of rectified varieties. Then for any $m \in \mathbb{N}$ there is $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ and any m-smooth function $g \in C_c^\infty(B_m^{\mathbf{X}_1,k})$, the function $f := \gamma_*(g)$ on $B_\infty^{\mathbf{X}_2,k}$ defined by*

$$f(y) = \sum_{x \in (\gamma^{-1}(y))(\mathbb{F}_{\ell^k}((t))) \cap B_m^{\mathbf{X}_1,k}} g(x)$$

is m'-smooth.

Proof. WLOG assume that g is real and non-negatively valued. We will use the criterion for m-smoothness (Lemma 5.2(1)). Fix m.

By Theorem 4.1 there is $m_1 > m$ such that

$$\forall k \in \mathbb{N}, x \in B_m^{\mathbf{X}_1,k} \text{ the map } \gamma|_{B_{-m_1}^{\mathbf{X}_1,k}(x)} \text{ is a monomorphism} \tag{5.1}$$

By Theorem 4.2 there exists $m' > m_1$ such that for any k and any $x \in B_m^{\mathbf{X}_1,k}$ we have

$$\gamma(B_{-m_1}^{\mathbf{X},k}(x)) \supset B_{-m'}^{\mathbf{Y},k}(\gamma(x)). \tag{5.2}$$

Fix $k \in \mathbb{N}$. For $y \in B_\infty^{\mathbf{X}_2,k}$, denote

$$\mathfrak{F}_y := (\gamma^{-1}(y))(\mathbb{F}_{\ell^k}((t))).$$

Fix $y \in B_\infty^{\mathbf{X}_2,k}$, and denote

$$\{x_1, \ldots, x_N\} := \mathfrak{F}_y \cap B_m^{\mathbf{X}_1,k}.$$

By (5.1) and Lemma 3.4(2), for every $i, j \in \{1, \ldots, N\}$, we have:

$$B_{-m_1}^{\mathbf{X}_1,k}(x_i) \cap B_{-m_1}^{\mathbf{X}_1,k}(x_j) = \emptyset. \tag{5.3}$$

Let $y' \in B_{-m'}^{\mathbf{X}_2,k}(y)$. We obtain

$$\gamma_*(g)(y') = \sum_{x \in \mathfrak{F}_{y'}} g(x) \overset{(5.2)}{\geq} \sum_{i=1}^{N} \sum_{x \in \mathfrak{F}_{y'} \cap B_{-m_1}^{\mathbf{X}_1,k}(x_i)} g(x) \overset{(5.3)}{\geq}$$

$$\geq \sum_{i=1}^{N} \min(g(B_{-m_1}^{\mathbf{X}_1,k}(x_i))) = \sum_{i=1}^{N} g(x_i) = \gamma_*(g)(y).$$

By Lemma 5.2(1) this implies the assertion. □

Theorem 5.7. *Let $\gamma : \mathbf{X}_1 \to \mathbf{X}_2$ be a smooth map of μ-rectified varieties. Then for any $m \in \mathbb{N}$ there is $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ and any m-smooth function $g \in C_c^\infty(B_\infty^{\mathbf{X}_1,k})$ there is an m'-smooth function $f \in C_c^\infty(B_{m'}^{\mathbf{X}_2,k})$ such that:*

$$\gamma_*(g\mu_m^{\mathbf{X}_1,k}) = f \cdot \mu_{m'}^{\mathbf{X}_2,k}.$$

Proof.

Case 1. γ is an open embedding:
Follows from Lemma 5.3 and Lemma 5.5.

Case 2. γ is étale, both rectifications are simple and the form on $\mathbf{X}_1$ is the pullback of the form on $\mathbf{X}_2$:
Follows from Lemma 5.6.

Case 3. γ is étale, and both rectifications are simple:
Follows from the previous 2 cases.

Case 4. γ is étale, and the rectification of $\mathbf{X}_2$ is simple:
Follows from the previous case.

Case 5. γ is étale and can be written as a composition: $\mathbf{X}_1 \to \mathbf{X}_3 \to \mathbf{X}_2$ where the rectification of $\mathbf{X}_3$ is simple and $\mathbf{X}_3 \to \mathbf{X}_2$ is an open embedding:
Follows from the previous case and Case 1.

Case 6. γ is étale and the cover of $\mathbf{X}_1$ is obtained of covers of the preimages of the covering sets on $\mathbf{X}_2$:
Follows from the previous case.

Case 7. γ is étale:
Follows from the previous case and Case 1.

Case 8. $\mathbf{X}_1 = \mathbf{X}_2 \times \mathbb{A}^n$, γ is the projection, the rectifications on $\mathbf{X}_i$ are simple, and the rectification of $\mathbf{X}_1$ is obtained from the rectification of $\mathbf{X}_2$ in the natural way:
In this case we can take $m' = m$. The assertion follows from the fact that for 2 points $x_1, x_2 \in B_\infty^{\mathbf{X}_2,k}$ with $x_2 \in B_m^{\mathbf{X}_2,k}(x_1)$ we have $j_1^*(g) = j_2^*(g)$ for any m-smooth $g \in C_c^\infty(B_\infty^{\mathbf{X}_1,k})$, where $j_i : \mathbb{A}^n \to \mathbf{X}_1$ are given by $j_i(a) = (x_i, a)$.

Case 9. $\mathbf{X}_1 = \mathbf{X}_2 \times \mathbb{A}^n$, γ is the projection, and $\mathbf{X}_2$ is affine:
Follows from the previous case and Case 1.

Case 10. The general case:
Follows from the previous case Case 7, Case 1 and the structure theorem for smooth maps ([Sta25, Theorem 039Q]).

□

6. Effectively surjective maps

In this section we introduced a version of surjectivity of a map between algebraic varieties. We complete the proof of Theorem G, and prove Theorem E - a criterion for effective surjectivity (See Theorem 6.3 below).

Definition 6.1. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a map of rectified varieties. We say that γ is effectively surjective iff for any m$\in \mathbb{N}$ there is $m' \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ we have*

$$\gamma(B_{m'}^{\mathbf{X},k}) \supset B_m^{\mathbf{Y},k}.$$

From Corollary 3.6 we obtain:

Lemma 6.2. *The property of a map $\gamma : \mathbf{X} \to \mathbf{Y}$ being effectively surjective does not depend of the rectifications of the varieties $\mathbf{X}$ and $\mathbf{Y}$.*

Theorem 6.3. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a smooth map of algebraic varieties that is onto on the level of points over any field. Then γ is effectively surjective.*

For this we will need some preparations.

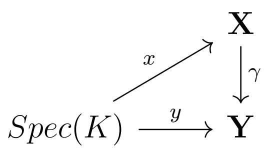
Lemma 6.4. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a map of algebraic varieties. Assume that γ is onto on the level of points for any field (that contains $\mathbb{F}_\ell$). Then, $\mathbf{Y}$ admits a stratification such that γ admits a section for each strata.*

Remark 6.5. *This is a standard result which is valid over any field. For completeness, we include its proof here.*

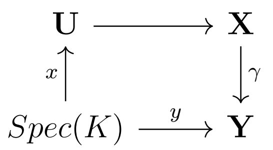
Proof. We will prove the statement by Noetherian Induction. So it is enough to show that γ admits a section on some nonempty open set $\mathbf{V} \subset \mathbf{Y}$. Without loss of generality we can assume that $\mathbf{Y}$ is irreducible and affine.

Case 1: $\dim(\mathbf{Y}) = 0$
obvious.

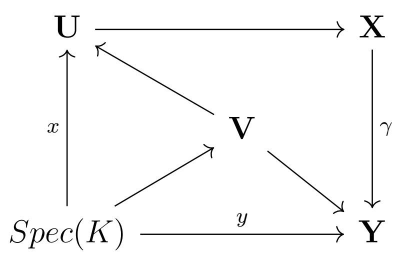
Case 2: $\dim(\mathbf{Y}) > 0$
Let K be the field of rational functions on $\mathbf{Y}$. By the assumption, $\gamma(K) : \mathbf{X}(K) \to \mathbf{Y}(K)$ is onto. We have a canonical point $y \in \mathbf{Y}(K)$. This gives us a point $x \in \mathbf{X}(K)$ such that $y = \gamma(x)$. So we get a commutative diagram:



We have an affine open set $\mathbf{U} \subset \mathbf{X}$ such that $x \in \mathbf{U}(K)$. In other words we get a diagram



This gives us a map $O_{\mathbf{U}}(\mathbf{U}) \to K$. Since $O_{\mathbf{U}}(\mathbf{U})$ is finitely generated over $\mathbb{F}_\ell$, the image of this map lies inside $f^{-1}O_{\mathbf{Y}}(\mathbf{Y})$ for some $f \in O_{\mathbf{Y}}(\mathbf{Y})$. Let $\mathbf{V} := \mathbf{Y}_f$ be the non vanishing locus of f. We get a commutative diagram:



This gives the requested section.

□

The following follows immediately from Corollary 3.6:

Lemma 6.6. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a map of algebraic varieties defined over $\mathbb{F}_\ell$ that admits a section. Then γ is effectively surjective.*

Corollary 6.7. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a smooth map of algebraic varieties defined over $\mathbb{F}_\ell$. Let $\mathbf{U} \subset \mathbf{Y}$ be open and $\mathbf{Z} := \mathbf{Y} \smallsetminus \mathbf{U}$. Assume that $\mathbf{Z}$ is smooth. Assume also that $\gamma|_{\gamma^{-1}(\mathbf{Z})} : \gamma^{-1}(\mathbf{Z}) \to \mathbf{Z}$ and $\gamma|_{\gamma^{-1}(\mathbf{U})} : \gamma^{-1}(\mathbf{U}) \to \mathbf{U}$ are effectively surjective. Then so is γ.*

Proof. Fix an integer m. Choose a rectification of $\gamma^{-1}(\mathbf{U})$ which is compatible with the rectification on $\mathbf{X}$ in the following sense: For any $m \in \mathbb{Z}, k \in \mathbb{N}$ we have

- $B_m^{\gamma^{-1}(\mathbf{U}),k} \subset B_m^{\mathbf{X},k}$.
- For any $x \in B_\infty^{\mathbf{X},k}$ we have $B_{-m}^{\gamma^{-1}(\mathbf{U}),k}(x) \subset B_{-m}^{\mathbf{X},k}(x)$.

By Corollary 3.8 there exists $m_1 > m$ such that for any k we have

$$B_{m_1}^{\mathbf{Z},k} \supset B_m^{\mathbf{Y},k} \cap B_\infty^{\mathbf{Z},k}. \tag{6.1}$$

By the assumption there exists $m_2 > m_1$ such that for any $k \in \mathbb{N}$ we have

$$\gamma(B_{m_2}^{\gamma^{-1}(\mathbf{Z}),k}) \supset B_{m_1}^{\mathbf{Z},k}. \tag{6.2}$$

By the Proposition 3.5 there exists $m_3 > m_2$ such that for any k we have

$$B_{m_3}^{\mathbf{X},k} \supset B_{m_2}^{\gamma^{-1}(\mathbf{Z}),k}. \tag{6.3}$$

By the Theorem 4.2 there exists $m_4 > m_3$ such that for any k and for any $x \in B_{m_3}^{\mathbf{X},k}$ we have

$$\gamma(B_{-m_3}^{\mathbf{X},k}(x)) \supset B_{-m_4}^{\mathbf{Y},k}(\gamma(x)). \tag{6.4}$$

By the Proposition 4.3 there exists $m_5 > m_4$ such that for any k we have

$$B_{-m_4}^{\mathbf{Y},k}(\mathbf{Z}) \cup B_{m_5}^{\mathbf{U},k} \supset B_{m_4}^{\mathbf{Y},k}. \tag{6.5}$$

By the assumption there exists $m_6 > m_5$ such that for any k we have

$$\gamma(B_{m_6}^{\gamma^{-1}(\mathbf{U}),k}) \supset B_{m_5}^{\mathbf{U},k}. \tag{6.6}$$

By Lemma 3.4(1,2) for any 2 integers $a, b \in \mathbb{N}$ we have:

$$\bigcup_{x \in B_a^{\mathbf{Y},k} \cap B_\infty^{\mathbf{Z},k}} B_{-b}^{\mathbf{Y},k}(x) = B_a^{\mathbf{Y},k} \cap B_{-b}^{\mathbf{Y},k}(\mathbf{Z}) \tag{6.7}$$

and

$$\bigcup_{x \in B_a^{\mathbf{X},k} \cap B_\infty^{\gamma^{-1}(\mathbf{Z}),k}} B_{-b}^{\mathbf{Y},k}(x) = B_a^{\mathbf{X},k} \cap B_{-b}^{\mathbf{Y},k}(\gamma^{-1}(\mathbf{Z})). \tag{6.8}$$

Take $m' = m_6$. For any $k \in \mathbb{N}$ we have

$$\begin{aligned}
\gamma(B^{\mathbf{X},k}_{m'}) &= \gamma(B^{\mathbf{X},k}_{m_6}) \overset{\text{Lemma 3.4(1)}}{\supset} \gamma((B^{\mathbf{X},k}_{-m_3}(\gamma^{-1}(\mathbf{Z})) \cap B^{\mathbf{X},k}_{m_3}) \cup B^{\gamma^{-1}(\mathbf{U}),k}_{m_6}) = \\
&= \gamma(B^{\mathbf{X},k}_{-m_3}(\gamma^{-1}(\mathbf{Z})) \cap B^{\mathbf{X},k}_{m_3}) \cup \gamma(B^{\gamma^{-1}(\mathbf{U}),k}_{m_6}) \supset \\
&\overset{(6.8,6.6)}{\supset} \gamma\left(\bigcup_{x \in B^{\mathbf{X},k}_{m_3} \cap B^{\gamma^{-1}(\mathbf{Z}),k}_{\infty}} B^{\mathbf{X},k}_{-m_3}(x)\right) \cup B^{\mathbf{U},k}_{m_5} = \\
&= \bigcup_{x \in B^{\mathbf{X},k}_{m_3} \cap B^{\gamma^{-1}(\mathbf{Z}),k}_{\infty}} \gamma(B^{\mathbf{X},k}_{-m_3}(x)) \cup B^{\mathbf{U},k}_{m_5} \supset \\
&\overset{(6.3)}{\supset} \bigcup_{x \in B^{\gamma^{-1}(\mathbf{Z}),k}_{m_2}} \gamma(B^{\mathbf{X},k}_{-m_3}(x)) \cup B^{\mathbf{U},k}_{m_5} \supset \\
&\overset{(6.4)}{\supset} \bigcup_{x \in B^{\gamma^{-1}(\mathbf{Z}),k}_{m_2}} B^{\mathbf{Y},k}_{-m_4}(\gamma(x)) \cup B^{\mathbf{U},k}_{m_5} = \\
&= \bigcup_{x \in \gamma(B^{\gamma^{-1}(\mathbf{Z}),k}_{m_2})} B^{\mathbf{Y},k}_{-m_4}(x) \cup B^{\mathbf{U},k}_{m_5} \supset \\
&\overset{(6.2)}{\supset} \bigcup_{x \in B^{\mathbf{Z},k}_{m_1}} B^{\mathbf{Y},k}_{-m_4}(x) \cup B^{\mathbf{U},k}_{m_5} \supset \\
&\overset{(6.1)}{\supset} \bigcup_{x \in B^{\mathbf{Y},k}_{m} \cap B^{\mathbf{Z},k}_{\infty}} B^{\mathbf{Y},k}_{-m_4}(x) \cup B^{\mathbf{U},k}_{m_5} \supset \\
&\overset{(6.7)}{\supset} (B^{\mathbf{Y},k}_{m} \cap B^{\mathbf{Y},k}_{-m_4}(\mathbf{Z})) \cup B^{\mathbf{U},k}_{m_5} \supset \\
&\supset B^{\mathbf{Y},k}_{m} \cap (B^{\mathbf{Y},k}_{-m_4}(\mathbf{Z})) \cup B^{\mathbf{U},k}_{m_5}) \supset \\
&\overset{(6.5)}{\supset} B^{\mathbf{Y},k}_{m} \cap B^{\mathbf{Y},k}_{m_4} = B^{\mathbf{Y},k}_{m}.
\end{aligned}$$

□

Proof of Theorem 6.3. By Lemma 6.4, there is a stratification $\mathbf{Y} = \bigcup_\alpha \mathbf{Y}_\alpha$ such that γ admits a section for each strata. By Lemma 6.6 the maps $\gamma : \gamma^{-1}(\mathbf{Y}_\alpha) \to \mathbf{Y}_\alpha$ are effectively surjective. We will show that $\gamma : \mathbf{X} \to \mathbf{Y}$ is effectively surjective by induction on the number of strata. The base follows from Lemma 6.6. For the induction step let $\mathbf{Y}_0$ be a closed stratum. Let $\mathbf{Y}' = \mathbf{Y} \smallsetminus \mathbf{Y}_0$. By the induction hypothesis, the map $\gamma^{-1}(\mathbf{Y}') \to \mathbf{Y}'$ is effectively surjective. By Lemma 6.6 the map $\gamma^{-1}(\mathbf{Y}_0) \to \mathbf{Y}_0$ is effectively surjective. So, by Corollary 6.7, γ is effectively surjective. □

Theorem 4.4 gives us the following:

Corollary 6.8. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a submersion of μ-rectified varieties. Assume that γ is effectively surjective. Then for any m there is m' such that* *for any $k \in \mathbb{N}$ we have*

$$\mu_m^{\mathbf{Y},k} < \ell^{km'} \gamma_*(\mu_{m'}^{\mathbf{X},k}).$$

Proof. Fix m. Since γ is effectively surjective there is m_1 such that for every $k \in \mathbb{N}$ we have

$$\gamma(B_{m_1}^{\mathbf{X},k}) \supset B_m^{\mathbf{Y},k}.$$

By Theorem 4.4, there is M such that for any k we have

$$\mu_m^{\mathbf{Y},k} \cdot 1_{\mathrm{Supp}(\gamma_*(\mu_{m_1}^{\mathbf{X},k}))} < \ell^{kM} \gamma_*(\mu_{m_1}^{\mathbf{X},k}).$$

Take $m' = m_1 + M$. For any $k \in \mathbb{N}$ we have

$$\mu_m^{\mathbf{Y},k} = \mu_m^{\mathbf{Y},k} \cdot 1_{B_m^{\mathbf{Y},k}} \leq \mu_m^{\mathbf{Y},k} \cdot 1_{\gamma(B_{m_1}^{\mathbf{X},k})} = \mu_m^{\mathbf{Y},k} \cdot 1_{\mathrm{Supp}(\gamma_*(\mu_{m_1}^{\mathbf{X},k}))} < \ell^{kM} \gamma_*(\mu_{m_1}^{\mathbf{X},k}) < \ell^{km'} \gamma_*(\mu_{m'}^{\mathbf{X},k}).$$

□

Index

$B_m^{\mathbf{X},k}$, $B_\infty^{\mathbf{X},k}$, 8
$\mu_m^{\mathbf{X},k}$, 9
□, 7
$|f|$, 8
$|\omega|$, 8
m-smooth, 22

effectively surjective, 25

rectification, rectified variety, 8

variety, 7

References

[AGKS] Avraham Aizenbud, Dmitry Gourevitch, David Kazhdan, and Eitan Sayag. The jet schemes of the nilpotent cone of $\mathfrak{gl}_n$ over $\mathbb{F}_\ell$ and analytic properties of the Chevalley map. Preprint available at https://www.wisdom.weizmann.ac.il/∼dimagur/.

[BKS24] Paul Breiding, Kathlén Kohn, and Bernd Sturmfels. *Metric algebraic geometry*, volume 53 of *Oberwolfach Seminars*. Birkhäuser/Springer, Cham, 2024.

[HC70] Harish-Chandra. *Harmonic analysis on reductive p-adic groups*, volume Vol. 162 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970. Notes by G. van Dijk.

[Kot05] Robert E. Kottwitz. Harmonic analysis on reductive p-adic groups and Lie algebras. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 393–522. Amer. Math. Soc., Providence, RI, 2005.

[KP13] Steven G. Krantz and Harold R. Parks. *The implicit function theorem*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2013. History, theory, and applications, Reprint of the 2003 edition.

[Mus01] Mircea Mustaţă. Jet schemes of locally complete intersection canonical singularities. *Invent. Math.*, 145(3):397–424, 2001. With an appendix by David Eisenbud and Edward Frenkel.

[Sta25] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2025.

[Wei21] Madeleine Weinstein. *Metric Algebraic Geometry*. PhD thesis, University of California, Berkeley, 2021.

Avraham Aizenbud, Faculty of Mathematical Sciences, Weizmann Institute of Science, 76100 Rehovot, Israel
Email address: aizenr@gmail.com
URL: https://www.wisdom.weizmann.ac.il/~aizenr/

Dmitry Gourevitch, Faculty of Mathematical Sciences, Weizmann Institute of Science, 76100 Rehovot, Israel
Email address: dimagur@weizmann.ac.il
URL: https://www.wisdom.weizmann.ac.il/~dimagur/

David Kazhdan, Einstein Institute of Mathematics, Edmond J. Safra Campus, Givaat Ram The Hebrew University of Jerusalem, Jerusalem, 91904, Israel
Email address: david.kazhdan@mail.huji.ac.il
URL: https://math.huji.ac.il/~kazhdan/

Eitan Sayag, Department of Mathematics, Ben Gurion University of the Negev, P.O.B. 653, Be'er Sheva 84105, ISRAEL
Email address: eitan.sayag@gmail.com
URL: www.math.bgu.ac.il/~sayage