SUMMARY FOR THE COURSE "INTRODUCTION TO REPRESENTATION THEORY", SPRING 2020

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1. Basic definitions and Schur's lemmas

Definition 1.1. *A **group** G is a set with a binary operation $G \times G \to G$, called multiplication, such that*

(1) $\forall f, g, h \in G . (fg)h = f(gh)$
(2) $\exists 1 \in G\, s.t. \forall g \in G,\, 1g = g1 = g$
(3) $\forall g \in G,\, \exists g^{-1} \in G\, s.t.\, gg^{-1} = g^{-1}g = 1$

*A **morphism of groups** $\phi : G \to H$ is a function $\phi : G \to H$ s.t. $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)\, \forall g_1, g_2 \in G$.*

Example 1.2. $\mathbb{Z}$ *- the group of integers,* $\mathbb{Z}/n\mathbb{Z}$ *= the cyclic group of order* n, $Sym(X)$*- the group of all bijections from* X *to itself. Also denoted by* Sym_n *or* S_n *if* X *has* n *elements. If* V *is a vector space of dimension* n *over a field* F *then we denote by* $GL(V)$ *or by* $GL(n, F)$ *the group of all invertible linear transformations from* V *to itself.*

Definition 1.3. *A G**-set** (a, X) is a set X together with a morphism of groups $a : G \to \mathrm{Sym}(X)$. We also say that G acts on X via a, and that a is an action of G on X. We will sometimes omit the a or the X from the notation. Also, we will sometimes write gx for $a(g)x$.*
*A **morphism of** G**-sets** $\nu : (a, X) \to (b, Y)$ is a function $\nu : X \to Y$ such that $\nu(a(g)x) = b(g)\nu(x), \forall g \in G, x \in X$.*
*Denote by X^G the **set of fixed points of** G in X, i.e. $X^G := \{x \in X : gx = x\, \forall g \in G\}$. For a point $x \in X$ denote by $G_x := Stab_G(x) := \{g \in G : gx = x\}$ the **stabilizer of** x in G and by $Gx := \{gx : g \in G\}$ the orbit of x.*
*An action of G on X is called **free** if all stabilizers are trivial and **transitive** if $Gx = X$ for some (and hence every) $x \in X$.*

Example 1.4.

(1) $\mathrm{Sym}(X)$ *acts on* X.
(2) $GL(V)$ *acts on* V.
(3) $G \times G$ *acts on* G *by* $(g_1, g_2) \cdot h = g_1 h g_2^{-1}$*. This gives rise to 3 actions of* G *on itself, corresponding to 3 embeddings of* G *to* $G \times G$*: left, right and diagonal.*

Definition 1.5. *Let H be a subgroup of G. Define an equivalence relation on G by $g_1 \sim g_2$ iff $g_1^{-1} g_2 \in H$. We will denote the set of equivalence classes by G/H and denote the equivalence class of g by gH. Then G/H has a natural action of G defined by $g_1(g_2 H) := (g_1 g_2)H$. We call it the **set of right** H**-cosets** in G.*

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*If the subgroup H is **normal**, i.e. satisfies $ghg^{-1} \in H \, \forall g \in G, h \in H$ then G/H has a natural group structure defined by $(g_1H)(g_2H) := g_1g_2H$.*

Proposition 1.6.
1. *$|G| = |G/H| \cdot |H|$, where $| \, |$ denotes the size of a set.*
2. *Any transitive G-set X is isomorphic the set of cosets G/G_x where $x \in X$ is any point.*
3. *Any G-set is a disjoint union of transitive G-sets (its orbits).*

Many important groups have natural actions that are straightforward from their definitions. Many theorems on groups and their subgroups come from actions of G on itself or on coset spaces G/H. G-sets are important, and one can use geometry to study them. However, one cannot "compute" in G-sets. In order to compute, one needs some algebraic structure, e.g. a vector space.

Definition 1.7. *A **representation** of a group G over a field F consists of a vector space V over F and a morphism of groups $\pi : G \to GL(V)$. We will denote the representation by (G, π, V) or (π, V) or π or V. The dimension of V is called the dimension of the representation. A one-dimensional representation is called a **character**. A **morphism of** representations $\phi : (\pi, V) \to (\tau, W)$ is a linear map $\phi : V \to W$ that is a morphism of G-sets, i.e. such that $\phi(\pi(g)v) = \tau(g)\phi(v), \forall g \in G, v \in V$.*

Here are some examples of characters.

Example 1.8.
1. *The trivial character (of any group): $\chi(g) = 1$ for all g.*
2. *The sign character of S_n (sign of permutation).*
3. *The determinant for $GL(n, F)$.*

Here are some examples of representations.

Example 1.9.
1. *The zero representation (of any group): $V = 0$, $GL(V)$ has one element.*
2. *$SO(2, \mathbb{R})$ acts on $\mathbb{R}^2$ by rotations.*
3. *$GL(V)$ acts on V.*
4. *$\mathrm{Sym}(X)$ acts on the space $F(X)$ of all functions $X \to F$.*

Exercise 1.10. *Let $\pi, \tau \in Rep(G)$ and let $\phi : \pi \to \tau$ be a morphism of representations which is an isomorphism of linear spaces. Show that ϕ is an isomorphism of representations. In other words, show that the linear inverse ϕ^{-1} is also a morphism of representations.*

Definition 1.11. *Let (π, V) and (τ, W) be representations of G (over the same field F). Define a representation of G on the **direct sum** $V \oplus W$ by $g(v, w) := (\pi(g)v, \tau(g)w)$.*

*Define a **dual** or **contragredient** representation (π^*, V^*) by*

$$(\pi^*(g)\phi)(v) := \phi(\pi(g^{-1})v).$$

*Let (σ, U) be a representation of H (over F). Define a representation of $G \times H$ on the **tensor product** $V \otimes U$ by $(g, h)(v \otimes u) := \pi(g)v \otimes \sigma(h)u$.*

In particular, if $G = H$ then $\pi \otimes \sigma$ is a representation of $G \times G$, which also becomes a representation of G using the diagonal embedding $\Delta : G \hookrightarrow G \times G$. This enables us to define an action of G on $\mathrm{Hom}_F(V, U) = V^* \otimes U$.

Exercise 1.12. *Check that* $\mathrm{Hom}_F(V, U)^G = \mathrm{Hom}_G(\pi, \sigma)$.

Definition 1.13. *A* ***subrepresentation*** *of* (G, π, V) *is a* G*-invariant subspace of* V*, with induced action of* G*.*

Example 1.14. *Any representation has (at least) 2 subrepresentations :* 0 *and* V*.*

Definition 1.15. *A representation is called* ***irreducible*** *if it has only 2 subrepresentations.*

Example 1.16.

(1) Any character is irreducible
(2) The action of $SO(2, \mathbb{R})$ *on* $\mathbb{R}^2$ *by rotations is irreducible, while the action of* $\mathbb{R}^\times$ *on* $\mathbb{R}^2$ *by homotheties is not.*

Exercise 1.17. *Every irreducible representation of a finite group is finite dimensional.*

In the next lecture we will show that every representation is a direct sum of irreducible ones, and for a given group there is a finite number of isomorphism classes of irreps (unlike prime numbers). Thus, the main goals of representation theory are to classify all irreducible representations of a given group (up to isomorphism) and given a representation to find its decomposition to irreducible ones.

The most important properties of irreducible representations are Schur's lemmas.

Lemma 1.18. *Let* ρ *and* σ *be irreps of a group* G*.*

(1) Any non-zero morphism $\phi : \rho \to \sigma$ *is an isomorphism.*
(2) If the field F *is algebraically closed and* ρ *is finite-dimensional then* $\mathrm{Hom}(\rho, \rho) = F \cdot Id$*.*

Proof. (1) $\mathrm{Ker}\,\phi$ is a subrepresentation of ρ and $\mathrm{Im}\,\phi$ is a subrepresentation of σ.
(2) Let $\varphi \in \mathrm{Hom}(\rho, \rho)$ and λ be an eigenvalue of φ. Since $\varphi - \lambda Id$ is not invertible, (1) implies that it is zero. □

Corollary 1.19. *Every irrep of a finite commutative group over an algebraically closed field is one-dimensional.*

Exercise 1.20. *Every irrep of a commutative group over* $\mathbb{R}$ *is at most 2-dimensional. Give an example of a 2-dimensional irrep.*

Exercise 1.21. *Let* $(\pi_1, V_1), (\pi_2, V_2)$ *be irreps of a group* G*. Consider the direct sum* (π, V) *of these representations. The space* V *has four* G*-invariant coordinate subspaces* $0, V_1, V_2, V$*. Show that the representations* π_1 *and* π_2 *are isomorphic if and only if there exists a non-coordinate* G*-invariant subspace in* V *(i.e. a subspace distinct from the four subspaces listed above).*

2. Existence and uniqueness of decomposition to irreducibles, intertwining numbers and the group algebra.

From now on we consider only finite groups.

Definition 2.1 (Exercise). *A representation π is called completely reducible if one of the following equivalent conditions holds.*

1. *π is a direct sum of irreducible representations.*
2. *For every subrepresentation $\tau \subset \pi$ there exists another subrepresentation $\tau' \subset \pi$ such that $\pi = \tau \oplus \tau'$.*

Note that an irreducible representation is completely reducible :-).

Theorem 2.2 (Weyl-Mashke). *Suppose that $|G|$ is not zero in F. Then every representation (π, V) of G over F is completely reducible.*

Proof. Let $\tau \subset \pi$. It is enough to find a G-invariant linear projection on τ. We take any linear projection on τ and average it. Namely, we take a linear map $p : V \to V$ s.t. $p^2 = p$ and $\operatorname{Im} p = \tau$ and replace it by $p' := |G|^{-1} \sum_{g \in G} \pi(g) p \pi(g^{-1})$. Check that $p'^2 = p'$, $\operatorname{Im} p' = \tau$ and p' is G-invariant. $\square$

The idea of averaging is very important. It always gives something G-invariant, but sometimes produces zero. It already takes advantage of linearity of our subject - we would not be able to do such a thing with G-sets.

The assumptions that G is finite and $|G|$ is not zero in F are necessary, as shown by the following example.

Example 2.3. *Define $A \in Mat_2(F)$ by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let the group $\mathbb{Z}$ act on F^2 by $\pi(n) := A^n$. Then this representation is not completely reducible.*
If $char F = p$ then the same example gives a representation of the finite group $\mathbb{Z}/p\mathbb{Z}$.

From now on we assume $char F = 0$ and F is algebraically closed. Also, we consider only finite-dimensional representations.

Corollary 2.4. *Any matrix A with $A^n = Id$ is diagonalizable.*

In order to prove uniqueness of the decomposition we introduce a very important notion, called intertwining number.

Notation 2.5. *We denote by $Rep(G)$ the collection of all representations of G and by $Irr(G)$ the set of isomorphism classes of irreducible representations of G. In the next lecture we will show that the set $Irr(G)$ is finite.*

Definition 2.6. *Let $\pi, \tau \in Rep(G)$. Define the* ***intertwining number*** *of π and τ by $\langle \pi, \tau \rangle := \dim \operatorname{Hom}_G(\pi, \tau)$.*

Lemma 2.7. *The "form" $\langle \cdot, \cdot \rangle$ is "bilinear and symmetric". Namely*

1. $\langle \pi_1 \oplus \pi_2, \tau \rangle = \langle \pi_1, \tau \rangle + \langle \pi_2, \tau \rangle$
2. $\langle \pi, \tau_1 \oplus \tau_2 \rangle = \langle \pi, \tau_1 \rangle + \langle \pi, \tau_2 \rangle$
3. $\langle \bigoplus a_i \pi_i, \bigoplus b_j \tau_j \rangle = \sum_i \sum_j a_i b_j \langle \pi_i, \tau_j \rangle$,
   where a_i and b_j are natural numbers or zeros.

(4) *If π_i are irreducible, and a_i and b_i are natural numbers or zeros then* $\langle\bigoplus a_i\pi_i, \bigoplus b_i\pi_i\rangle = \sum_i a_ib_i$
(5) $\langle\pi,\tau\rangle = \langle\tau,\pi\rangle$

Proof. (1)-(2) are obvious and imply (3), which in turn implies (4) using Schur's lemmas. Complete reducibility + (4) implies (5). □

Note that we just proved that the spaces $\mathrm{Hom}_G(\pi,\tau)$ and $\mathrm{Hom}_G(\tau,\pi)$ are equidimensional and hence isomorphic, but we have no natural isomorphism between them.

Corollary 2.8. *The decomposition of any representation to a direct sum of irreducible ones is unique. The multiplicity with which an irrep σ appears in a representation π equals $\langle\sigma,\pi\rangle$.*

Corollary 2.9. *A representation π is irreducible if and only if $\langle\pi,\pi\rangle = 1$.*

For a vector space V denote $\mathrm{End}(V) := \mathrm{Hom}(V,V)$. Note that $\mathrm{End}(V) = V \otimes V^*$. Thus, let us study some properties of actions on tensor products.

Let $\pi \in Rep(G)$ and $\tau \in Rep(H)$.

Exercise 2.10. *Show that $(\pi\otimes\tau)|_G = (\dim\tau)\pi$ and $(\pi\otimes\tau)|_H = (\dim\pi)\tau$.*

Notation 2.11. *For a representation V denote by V^G the space of G-invariant vectors.*

Exercise 2.12. *Show that $(\pi\otimes\tau)^{G\times H} = \pi^G \otimes \tau^H$.*

Lemma 2.13. *Let $\rho \in Irr(G)$ and $\sigma \in Irr(H)$. Then $\rho\otimes\sigma \in Irr(G\times H)$.*

Proof.

$$\mathrm{End}_{G\times H}(\rho\otimes\sigma) = (\mathrm{End}_F(\rho\otimes\sigma))^{G\times H} = (\rho^*\otimes\sigma\otimes\sigma^*\otimes\rho)^{G\times H} = (\rho^*\otimes\rho\otimes\sigma\otimes\sigma^*)^{G\times H} =$$
$$(\rho^*\otimes\rho)^G\otimes(\sigma\otimes\sigma^*)^H = \mathrm{End}_F(\rho)^G\otimes\mathrm{End}_F(\sigma)^H = \mathrm{End}_G(\rho)\otimes\mathrm{End}_H(\sigma).$$

Thus, $\langle\rho\otimes\sigma,\rho\otimes\sigma\rangle = \langle\rho,\rho\rangle\langle\sigma,\sigma\rangle = 1$. □

Exercise 2.14. *Prove that every irrep of $G\times H$ can be obtained in this way.*

Corollary 2.15. *If $\rho \in Irr(G)$ then $\mathrm{End}_F(\rho) \in Irr(G\times G)$.*

Definition 2.16 (Group algebra)**.** *Define the **group algebra** $\mathcal{A}(G)$ of G to be the algebra spanned over F by the symbols δ_g, $g\in G$ with multiplication defined by $\delta_g\delta_h = \delta_{gh}$. Note that this is an associative non-commutative (unless G is commutative) algebra with unit (equal to δ_1). We can also view it as the algebra of functions from G to F, or the algebra of measures on G, with multiplication given by convolution:*

$$f * h(g) := \sum_{x\in G} f(gx^{-1})h(x)$$

*We define a representation of $G\times G$ on $\mathcal{A}(G)$ by $(g_1,g_2)\delta_x := \delta_{g_1xg_2^{-1}}\,\forall x\in G$ or, equivalently, $((g_1,g_2)f)(x) := f(g_1^{-1}xg_2)\,\forall f\in\mathcal{A}(G)$, $x\in G$. This representation is called the **regular representation** of G. Its restrictions on first and second coordinate of $G\times G$ are called the **left regular** and **right regular** representations respectively.*

Definition 2.17. *A representation of an algebra with unit A on a vector space V is a morphism of algebras with unit $A\to\mathrm{End}(V)$.*

Exercise 2.18. *A representation (π, V) of G defines a representation of $\mathcal{A}(G)$ on V and vice versa.*

Lemma 2.19. *If $\rho \in Irr(G)$ then the natural morphism of algebras $\mathcal{A}(G) \to \operatorname{End}_F(\rho)$ is onto.*

Proof. $\operatorname{End}_F(\rho)$ is an irrep of $G \times G$ and the image of this morphism is a non-zero subrepresentation. $\square$

3. Decomposition of the regular representation. Corollaries on number and dimensions of irreducible representations. Examples for small symmetric groups

Lemma 3.1. *Let V be a vector space. Then $\langle A, B\rangle := \operatorname{Tr}(AB)$ defines a non-degenerate symmetric bilinear form on $\operatorname{End}(V)$. Moreover, if V is a representation of G then this form is invariant with respect to the diagonal action of G. This form is called the* ***trace form****.*

Theorem 3.2. *The natural morphism*

$$\phi : \mathcal{A}(G) \to \prod_{\sigma \in Irr(G)} \operatorname{End}_F(\sigma)$$

given on each coordinate σ by

$$\phi_\sigma(f) := \sigma(f) = \sum_{g \in G} f(g)\sigma(g)$$

is an isomorphism of algebras and of representations of $G \times G$.

Proof.
1. It is easy to see that ϕ is a morphism of algebras and of representations of $G \times G$. Thus it is enough to show that ϕ is one to one and onto.
2. Suppose $f \in \operatorname{Ker} \phi \subset \mathcal{A}(G)$. Then f acts by zero on any irreducible representation of G and thus on any representation of G. Thus, f acts by zero on $\mathcal{A}(G)$, but $f\delta_1 = f$ and thus $f = 0$.
3. Define a morphism $\psi : \bigoplus_{\sigma \in Irr(G)} \operatorname{End}_F(\sigma) \to \mathcal{A}(G)$ in the following way. For $A \in \operatorname{End}(\sigma)$ let $\psi(A)(g) := \operatorname{Tr}(\sigma(g^{-1})A)$, and continue by linearity to the direct sum. Let us show that the composition $\phi\psi$ is an embedding of $\bigoplus_{\sigma \in Irr(G)} \operatorname{End}_F(\sigma)$ into $\prod_{\sigma \in Irr(G)} \operatorname{End}_F(\sigma)$.

   Indeed, let $\sigma, \rho \in \operatorname{Irr}(G)$ and let $\phi\psi^\rho_\sigma : \operatorname{End}_F(\sigma) \to \operatorname{End}_F(\rho)$ be the projection to $\operatorname{End}_F(\rho)$ of the restriction to σ of $\phi\psi$. Since $\phi\psi^\rho_\sigma$ is a morphism of representations of $G \times G$, that are irreducible by Corollary 2.15, Schur's lemma implies that it is constant if $\sigma \simeq \rho$ and zero otherwise. The constant is non-zero, since ϕ is an embedding, and $\psi|_{\operatorname{End}_F(\sigma)}$ is an embedding by Lemmas 2.19 and 3.1. Thus $\phi\psi$ is a direct sum of embeddings, and thus is an embedding.

   Thus ψ is an embedding. This implies that $\operatorname{Irr} G$ is a finite set.
4. Now, by (3) the R.H.S. is finite dimensional and its dimension is at most the dimension of L.H.S, and by (2), ϕ is one to one. Thus ϕ is an isomorphism.

$\square$

Corollary 3.3. *(1) $Irr(G)$ is finite and*

$$\sum_{\sigma_{Irr(G)}} (\dim \sigma)^2 = |G|.$$

(2) $|Irr(G)|$ equals the number of conjugacy classes in G.

Proof. (1): both are equal to the dimension of $\mathcal{A}(G)$.
(2): both are equal to the dimension of the center of $\mathcal{A}(G)$. □

Example 3.4. *If G is commutative then $|Irr(G)| = |G|$ and all irreps are characters.*

This example and Theorem 3.2 show that representation theory is in some sense Fourier analysis on non-commutative groups.

The main goal of representation theory is to classify all irreducible representations of a given group G. For complex representations of simple finite groups this has been achieved in 20th centaury, together with the classification of such groups.

We will now consider small symmetric groups, but first we will prove an important general lemma.

Lemma 3.5. *Let X and Y be G-sets. Then $\langle F(X), F(Y)\rangle$ equals the number of orbits of G in $X \times Y$ under the diagonal action.*

Proof. Using the basis of δ-functions on $F(X)$ we have a natural isomorphism

$$\mathrm{Hom}_F(F(X), F(Y)) \cong F(X \times Y)$$

as representations of $G \times G$, and thus as representations of ΔG. Thus

$$\mathrm{Hom}_G(F(X), F(Y)) = \mathrm{Hom}_F(F(X), F(Y))^{\Delta G} \cong F(X \times Y)^{\Delta G} \cong F((X \times Y)/\Delta G),$$

where $(X \times Y)/\Delta G$ is the set of ΔG-orbits. Thus the dimension is the number of orbits. □

Definition 3.6. *X is called double-transitive if ΔG has 2 orbits on $X \times X$.*

In other words, for any 2 pairs $(x, y), (x', y') \in X \times X$ with $x \neq y$ and $x' \neq y'$ there exists $g \in G$ s.t. $gx = x'$ and $gy = y'$.

Corollary 3.7. *If the action of G on X is double-transitive then $F_0(X)$ is irreducible.*

Proof. $F(X) = F \oplus F_0(X)$, thus if $\langle F(X), F(X)\rangle = 2$ then $\langle F_0(X), F_0(X)\rangle = 1$. □

Example 3.8. *Classification of $Irr(S_2)$.*
S_2 acts on $\{1, 2\}$. We have decomposition to irreps $F(\{1, 2\}) = F \oplus F_0(\{1, 2\})$. Since S_2 is of size 2, we found all irreps. In fact, $F_0(\{1, 2\})$ is the sign character.

Example 3.9. *Classification of $Irr(S_3)$.*

S_3 acts double-transitively on $\{1, 2, 3\}$. Thus $\rho := F_0(\{1, 2, 3\})$ is irreducible. We also have the trivial representation and the sign character. Since S_3 has 3 conjugacy classes, this is all. We can also check this using dimensions: $2^2 + 1^2 + 1^2 = 3!$.

Example 3.10. *Classification of $Irr(S_4)$.*
S_4 acts double-transitively on $X := \{1, 2, 3, 4\}$. Thus $\rho := F_0(\{1, 2, 3, 4\})$ is irreducible. We also have the trivial representation and the sign character. S_4 has 5 conjugacy classes, thus there are two more irreps. The sums of squares of their dimensions is $4!-1^2-1^2-3^2 = 13$. Thus the dimensions of the missing irreps are 2 and 3.

Let Y be the set of subsets of X of size 2. Then S_4 has 3 orbits on $Y \times Y$: the diagonal orbit, pairs with one common element, and pairs without common elements. Geometrically, X can be viewed as vertices of a tetrahedron, and Y as its edges.
Thus, by the lemma, $F(Y)$ is the sum of 3 non-isomorphic irreps.
One of them is the trivial rep. Is ρ inside? To see this we compute $\langle F(X), F(Y)\rangle$. Since S_4 has has two orbits on $X \times Y$, $\langle F(X), F(Y)\rangle = 2$ and they have 2 irreps in common.
Since $F(X) = F \oplus \rho$, we get that $F(Y)$ includes ρ with multiplicity one.
Thus $F(Y) = F \oplus \rho \oplus \tau$, where τ is a 2-dimensional irrep.
We miss a 3-dimensional irrep. I claim that it is $\operatorname{sgn} \cdot \rho$. We just need to check that it is not isomorphic to ρ. We can check this on a simple permutation (12), using trace.
Indeed, the trace of the action of (12) on $F(X)$ is 2, and thus $\operatorname{Tr} \rho(12) = 1$. On the other hand, $\operatorname{Tr}(\operatorname{sgn}(12) \cdot \rho(12)) = -1$, and thus these reps are not isomorphic.

Exercise 3.11. $\tau \cong \operatorname{sgn} \cdot \tau$

4. Isotypic components; Characters, Schur orthogonality relations

4.1. Isotypic components.

Definition 4.1. *A representation is called **isotypic** if it is a direct sum of isomorphic irreducible representations.*

Exercise 4.2. *The following are equivalent:*

(1) π is isotypic
(2) All irreducible subrepresentations of π are isomorphic
(3) If $\pi \simeq \omega \oplus \tau$ with $\langle \omega, \tau\rangle = 0$ then either $\omega = 0$ or $\tau = 0$.

Theorem 4.3. *Let $(\pi, V) \in Rep(G)$. Then there exists a unique set of subrepresentations V_i such that $V = \bigoplus_{i=1}^k V_i$, V_i are isotypic, and $\langle V_i, V_j\rangle = 0$. Moreover, for any subrepresentation $W \subset V$, we have $W = \bigoplus_{i=1}^k (W \cap V_i)$.*

Proof. By induction. Existence is easy. Uniqueness follows from the "moreover" part. To prove the "moreover" part, fix a decomposition $V = \bigoplus V_i$, let $W \subset V$ and consider the decomposition $W = \bigoplus W_i$ where W_i has the same type as V_i, or is zero. Then $W \cap V_i \subset W_i$. On the other hand, W_i has zero projection on V_j, for $j \neq i$ and thus $W_i \subset V_i$. Thus $W_i = V_i \cap W$. $\square$

The V_i are called the isotypic components of π.

Definition 4.4. *If all isotypic components of π are irreducible then π is called multiplicity free.*

Lemma 4.5 (Easy)**.** *Every intertwining operator $L \in \operatorname{Hom}_G(\pi, \pi)$ preserves each isotypic component. In particular, if π is multiplicity free then L is scalar on each V_i.*

Exercise 4.6. *Barak has got a game for his birthday. In the game there was a cube with digits 1,...,6 on its faces, distributed somehow, not in the standard way. Each time he played with his friends and lost, he blamed the cube and modified it by replacing the number on every face by the average of the numbers written on the 4 neighbors of the face during the game round. What numbers will be written on the faces after 10 losses?*

Solution. Let V denote the 6-dimensional space of functions on the set X of faces of the cube and L denote the "averaging on neighbors" operator. Of course, we can guess that the answer will be approximately the constant function 3.5. However, to know how precise this approximation is we will need to diagonalize L and representation theory will help us.

Let G denote the group of motions of the cube and consider V as its representation. Then G has 3 orbits on X, thus $\langle V, V\rangle = 3$ and thus V is a sum of 3 non-isomorphic irreducible representations. One is, of course, the 1-dimensional space V_1 of constant functions. The other is the 2-dimensional space V_2 of "symmetric" functions with zero sum, namely functions that have the same value on opposite faces (and zero sum). The third is the 3-dimensional space V_3 of "anti-symmetric" functions.

The operator L commutes with the group action and thus acts by a scalar λ_i on each V_i. Taking convenient vectors from each V_i we get $\lambda_1 = 1, \lambda_2 = -1/2, \lambda_3 = 0$. Note that V has the natural form $\langle f, g\rangle := \sum f(x)\overline{g(x)}$, which is G-invariant and thus can be used to compute projections to V_i. Let ξ be the original function given by $(1, 2, 3, 4, 5, 6)$. Then its projection ξ_1 to V_1 is the constant function 3.5. The length of the projection to V_2 is at most $\sqrt{2((3.5-1)^2 + (3.5-2)^2 + (3.5-3)^2)} = \sqrt{17.5}$ and thus $|L^{10}(\xi) - \xi_1| \leq \sqrt{17.5}/2^{10} < 0,005$. □

Exercise 4.7. *Classify all irreducible representations of the group G from the solution of the last exercise.*

Hint. Use the action of G on faces, edges, vertices and main diagonals of the cube, and on regular tetrahedra inscribed in the cube.

4.2. **Characters.**

Definition 4.8. *Let $(\pi, V) \in Rep(G)$. Define a function χ_π on G by $\chi_\pi(g) := \operatorname{Tr}\pi(g)$.*

Lemma 4.9.

(1) If $\pi \backsimeq \tau$ then $\chi_\pi = \chi_\tau$.
(2) $\chi_\pi(hgh^{-1}) = \chi_\pi(g)$, i.e. $\chi_\pi \in Z(\mathcal{A}(G))$.
(3) $\chi_{\pi\oplus\tau} = \chi_\pi + \chi_\tau$.
(4) $\chi_{\pi\otimes\tau} = \chi_\pi\chi_\tau$.
(5) $\chi_\pi(g^{-1}) = \chi_{\pi^*}(g)$.

This lemma immediately follows from the corresponding properties of trace.

Definition 4.10. *Define a bilinear form on $\mathcal{A}(G)$ by*

$$\langle f, h\rangle := |G|^{-1}\sum_{g\in G} f(g)h(g^{-1})$$

Exercise 4.11. *This form is bilinear, symmetric and non-degenerate.*

4.3. **Schur orthogonality relations.**

Theorem 4.12 (Schur orthogonality relations)**.**

$$\langle \chi_\pi, \chi_\tau \rangle = \langle \pi, \tau \rangle$$

Proof. Let us first prove for the case when π is the trivial representation. Then $\mathrm{Hom}(\pi, \tau) = \tau^G$. Define $p : \tau \to \tau^G$ by $p := 1/|G| \sum \tau(g)$. Then $\mathrm{Im}\, p = \tau^G$ and $p|_{\tau^G} = Id$, i.e. p is a projection on τ^G. Thus, $\dim \tau^G = \mathrm{Tr}(p)$. On the other hand,

$$\mathrm{Tr}(p) = 1/|G| \sum \mathrm{Tr}(\tau(g)) = 1/|G| \sum_{g \in G} \chi_\tau(g) = 1/|G| \sum \chi_\pi(g^{-1})\chi_\tau(g) = \langle \chi_\pi, \chi_\tau \rangle$$

Now we will repeat the same argument for the general case, using the following exercise.

Exercise 4.13. *Let L, V be finite-dimensional linear spaces and let $X \in \mathrm{End}\, V, Y \in \mathrm{End}\, L$. Define $\Psi_{X,Y} : \mathrm{Hom}(L, V) \to \mathrm{Hom}(L, V)$ by $\Psi_{X,Y}(A) := XAY$. Then $\mathrm{Tr}\, \Psi_{X,Y} = \mathrm{Tr}\, X \,\mathrm{Tr}\, Y$.*

Hint There are (at least) two ways to solve this:
1) There is a "free' proof with tensor calculus.
2) In coordinates, $(YE_{ij}X)_{ij} = Y_{ii}X_{jj}$.

Now, let V be the space of π and L be the space of τ. Then $\mathrm{Hom}_G(\pi, \tau) = \mathrm{Hom}(V, L)^G$. For any $g \in G$ define $Q(g) : \mathrm{Hom}(V, L) \to \mathrm{Hom}(V, L)$ by $Q(g)(A) := \tau(g)A\pi(g^{-1})$. Then $1/|G| \sum_{g \in G} Q(g)$ is a projector from $\mathrm{Hom}(V, L)$ onto $\mathrm{Hom}_G(\pi, \tau) = \mathrm{Hom}(V, L)^G$. Thus

$$\langle \pi, \tau \rangle = \dim \mathrm{Hom}_G(\pi, \tau) = \mathrm{Tr}(1/|G| \sum_{g \in G} Q(g)) = 1/|G| \sum_{g \in G} \chi_\tau(g)\chi_\pi(g^{-1}) = \langle \chi_\pi, \chi_\tau \rangle$$

□

Corollary 4.14. *The character is a full invariant of a representation.*

Proof. $\pi = \bigoplus_{\rho \in IrrG} m_\rho \rho$, and m_ρ are determined by $m_\rho = \langle \pi, \rho \rangle = \langle \chi_\pi, \chi_\rho \rangle$. □

Corollary 4.15. *Characters of irreducible representations form an orthonormal basis for $Z(\mathcal{A}(G))$.*

Proof. By Lemma 4.9, characters of irreducible representations belong to $Z(\mathcal{A}(G))$. By the theorem and Schur's lemmas, they form an orthonormal set. By Corollary 3.3 their number is equal to $\dim Z(\mathcal{A}(G))$. Thus, they form an orthonormal basis. □

Lemma 4.16. *If $F = \mathbb{C}$ then $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$. Thus, on $Z(\mathcal{A}(G))$ the form $\langle\, ,\, \rangle$ coincides with the scalar product defined by $\langle f, h \rangle' = |G|^{-1} \sum_{g \in G} f(g)\overline{h(g)}$.*

Proof. As we showed some time ago, π has an invariant scalar product and thus $\pi^* \simeq \overline{\pi}$. Now, $\chi_\pi(g^{-1}) = \chi_{\pi^*}(g) = \chi_{\overline{\pi}}(g) = \overline{\chi_\pi(g)}$. □

4.4. **Dimensions of irreps divide the order of the group.**

Proposition 4.17. *Let $\rho \in Irr(G)$ and let $z_\rho = (\dim \rho / |G|) \sum_{g \in G} \chi_\rho(g^{-1})\delta_g$. Then $\rho(z_\rho) = Id$ and $\sigma(z_\rho) = 0$ for any $\sigma \not\simeq \rho \in Irr(G)$.*

Proof. Let $\omega \in Irr(G)$. Then, by the second Schur's lemma, $\omega(z_\rho)$ is a scalar. Now, $\operatorname{Tr} \omega(z_\rho) = \dim \rho / |G| \sum_{g \in G} \chi_\rho(g^{-1}) \chi_\omega(g) = \dim \rho \cdot \langle \rho, \omega \rangle$. Thus, $\omega(z_\rho) = Id$ if $\rho \simeq \omega$ and $\omega(z_\rho) = 0$ otherwise. $\square$

Corollary 4.18. *The inverse of the map $\mathcal{A}(G) \simeq \bigoplus_{\rho \in Irr(G)} \operatorname{End}_F(\rho)$ is given on the coordinate $\operatorname{End}_F(\rho)$ by $T \mapsto f_T(g) = (\dim \rho / |G|) \operatorname{Tr}(T \rho(g^{-1}))$.*

Corollary 4.19. *$\forall \rho \in Irr(G)$, $\dim \rho$ divides $|G|$.*

For the proof we will need

Definition 4.20. *A **lattice** is an abelian group without torsion.*

Theorem 4.21 (from commutative algebra)**.** *Any finitely generated lattice L has a basis, i.e. $L \simeq \mathbb{Z}^n$. In other words, $\exists l_1, ..., l_n \in L$ s.t. $\forall l \in L$, $\exists! \{a_i\}$ s.t. $l = \sum a_i l_i$, $l_i \in \mathbb{Z}$.*

Lemma 4.22. *Let V be a vector space over $\mathbb{Q}$, and let $L < V$ be a finitely generated lattice. Let $T : V \to V$ s.t. $T(L) \subset L$. Suppose that $T^2 = qT$. Then $q \in \mathbb{Z}$.*

Proof. Fix a basis $(l_1, ..., l_n)$ for L. Take $x \in L$ and let $y := Tx$. Then $Ty = qy$ and $T^k y = q^k y \, \forall k \geq 1$. Thus q is rational, and any power of the denominator of q divides all the coordinates of y. Thus $q \in \mathbb{Z}$. $\square$

We note that V can be infinite-dimensional.

Proof of Corollary 4.19. Apply the previous lemma to the following setting: $V := \mathcal{A}(G)$, vieweed as a vector space over $\mathbb{Q}$, $T :=$ convolution with $\sum \chi_\rho(g^{-1})\delta_g$, $q = |G| / \dim \rho$ and

$$L := \text{ lattice generated by } \{\xi \delta_g : \xi \text{ is a root of unity of order } |G|\}.$$

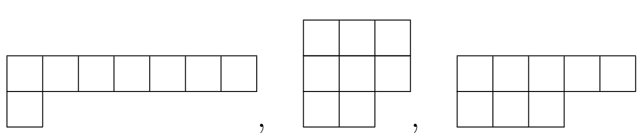
Let us show that the conditions of Lemma 4.22 are satisfied. We have $T/q = z_\rho$, and by Corollary 4.18, $z_\rho^2 = z_\rho$. Thus $(T/q)^2 = T/q$ and $T^2 = qT$. Next, for any $g \in G$, we have $\rho(g)^{|G|} = 1$, and thus all the eigenvalues of $\rho(g)$ are roots of unity of order $|G|$. Thus, for any g, $\chi_\rho(g)$ is an integer combination of roots of unity, and thus $z_\rho \in L$. It is easy to see that L is closed under convolution, and thus $T(L) \subset L$. $\square$

5. Classification of representations of symmetric groups

Let X be a set of size n and $G = \operatorname{Sym}(X) = S_n$.

Lemma 5.1. *Conjugacy classes in S_n = partitions of n, i.e. sets $(\alpha_1, ..., \alpha_k)$ of natural numbers s.t. $\alpha_1 + ... + \alpha_k = n$ and $\alpha_1 \geq ... \geq \alpha_k$.*

One can draw partitions using Young diagrams. That is, we draw a figure of boxes, in which raw i consists of α_i boxes.

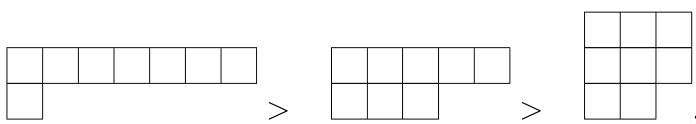
Example 5.2. 

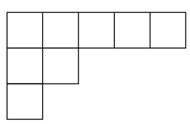
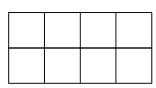
Let us now find an irreducible representation for each partition $\alpha = (\alpha_1, ..., \alpha_k)$. Denote by X_α the set of all decompositions of the set X to subsets $X_1, .., X_k$ s.t. $|X_i| = \alpha_i$. Each such decomposition can be pictured by a Young tableau. A Young tableau is a distribution of numbers $1 \dots n$ into a Young diagram such that the numbers in each row are increasing (some sources give different meaning to this word).

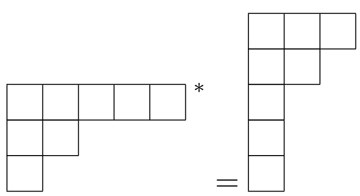
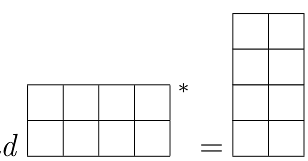
Example 5.3. $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&7&8\\\hline 5&6\\\cline{1-2}\end{array}$, $\begin{array}{|c|c|c|}\hline 1&5&6\\\hline 2&3&8\\\hline 4&7\\\cline{1-2}\end{array}$, $\begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline 6&7&8\\\cline{1-3}\end{array}$, $\begin{array}{|c|c|c|c|c|}\hline 1&2&6&7&8\\\hline 3&4&5\\\cline{1-3}\end{array}$

Definition 5.4. $T_\alpha := F(X_\alpha), \quad T'_\alpha := sgn \cdot T_\alpha$.

Introduce a partial ordering on partitions by $\lambda \leq \mu$ iff $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i \, \forall 1 \leq j \leq n$. Graphically, one obtains smaller partitions by moving boxes down.

Example 5.5. 

while  *and*  *are incomparable.*

Example 5.6.  *and* .

Definition 5.7. *Denote by α^* the transposed partition given by $\alpha_i^* := |\{j : \alpha_j \geq i\}|$.*

Exercise 5.8. *(1) α^* is a partition and $(\alpha^*)^* = \alpha$.*
(2) $\alpha \leq \beta \Leftrightarrow \alpha^* \geq \beta^*$.

Graphically, one obtains α^* by exchanging the rows with the columns.

Lemma 5.9 (Exc). *If X, Y are finite G-sets and χ is a character of G then the intertwining number $\langle \pi_X, \chi\pi_Y \rangle$ equals to the number of G-orbits O under the diagonal action on the set $X \times Y$ such that for any point $z \in O$, the restriction $\chi|_{G_z}$ of χ to the stabilizer G_z of z is trivial.*

Proof. As in the case $\chi = 1$ that was proved earlier,

$$\langle \pi_X, \chi\pi_Y \rangle = \dim \mathrm{Hom}_G(\pi_X, \pi_Y) = \dim F(X \times Y)^G,$$

where the action of G is by $f^g(x, y) = \chi(g^{-1}) f(g^{-1}x, g^{-1}y)$. Let $z = (x, y) \in X \times Y$. If χ is not trivial on G_z then every $f \in F(X \times Y)^G$ vanishes on z. If χ is trivial on G_z, then there exists a unique function $\delta_{Gz} \in F(X \times Y)^G$ that vanishes outside the orbit of z, and on the orbit of z is given by $\delta_{Gz}(gz) = \chi(g^{-1})$. These functions form a basis for $F(X \times Y)^G$, and thus the number of orbits with trivial restriction $\chi|_{G_z}$ equals $F(X \times Y)^G$ which in turn equals $\langle \pi_X, \chi\pi_Y \rangle$. $\square$

Theorem 5.10.
$$\langle T_\alpha, T'_\beta\rangle = \begin{cases} 0, & \alpha \not\leq \beta^*; \\ 1, & \alpha = \beta^*. \end{cases}$$

By the lemma, the theorem is equivalent to computation of the number of G-orbits on $X_\alpha \times X_\beta$ such that the sgn is trivial on the centralizer of any point of the orbit. We leave this computation as a difficult combinatorial exercise.

The theorem implies that T_α and T'_α have a unique joint irreducible component U_α and that these components are different for different α. This gives a classification of all irreducible representations of S_n. This classification is not very satisfying, but a long and detailed study of the intertwining operator of T_α and T'_α will lead to a (quite long) expression for the character of U_α. We will give here a formula for $\dim U_\alpha$, that we will prove later using Gelfand pairs:
$$\dim U_\alpha = \frac{n! \prod_{i<j}(l_i - l_j)}{l_1!...l_k!},$$
where $l_i = \alpha_i + k - i$, $i = 1, ..., k$.

6. Commutative groups: Fourier transform.

Let G be a finite commutative group. Then, by the second Schur's lemma all irreducible representations are 1-dimensional (characters). Their number is equal to $|G|$. Actually, the characters form a group: $(\chi \cdot \psi)(g) := \chi(g)\psi(g)$. It is called the (Pontryagin) dual group $\widehat{G}$. This group is not canonically isomorphic to G, but $G \approxeq \widehat{\widehat{G}}$ canonically.

Now, we constructed an isomorphism $\mathcal{A}(G) \approxeq \bigoplus \mathrm{End}(\sigma)$. For commutative G it becomes $\mathcal{F} : \mathcal{A}(G) \cong F(\widehat{G})$, where the multiplication in $F(\widehat{G})$ is pointwise. It is called Fourier transform. To see why let us write the explicit formula.
$$\mathcal{F}(f)(\chi) = \sum_{g\in G} f(g)\chi(g)$$
By Schur orthogonality relations, we know that the characters form an orthonormal basis for $\mathcal{A}(G)$ and thus f can be reconstructed from $\mathcal{F}(f)$ by
$$f(g) = |G|^{-1} \sum_{\chi \in \widehat{G}} \mathcal{F}(f)(\chi)\chi(g)^{-1}$$
since $\mathcal{F}(f)(\chi)$ is exactly the χ^{-1}-coordinate of f. This formula is called Fourier inversion formula. It also shows that $\mathcal{F}(\mathcal{F}(f))(g) = |G|^{-1} f(g^{-1})$, under the identification $G \approxeq \widehat{\widehat{G}}$.

To make things more familiar, let take $F = \mathbb{C}$. Then we have $\chi^{-1} = \overline{\chi}$. Let us consider $G = \mathbb{Z}/n\mathbb{Z}$ and choose a non-trivial character ψ by $\psi(k) := exp(\frac{2\pi i k}{n})$. Then for $c \in \mathbb{Z}/n\mathbb{Z}$ we have another character is given by $a \mapsto \psi(ca)$, and all characters of G are of this form. This gives an identification of G with $\widehat{G}$ and the familiar formulas for Fourier transform. The same thing happens for $G = \mathbb{R}$, but analysis comes in. For $G = S^1$, $\widehat{G} = \mathbb{Z}$ and Fourier transform becomes Fourier series.

Application. Multiplication of numbers.

Remark. The isomorphism $\mathcal{A}(G) \approxeq \bigoplus \mathrm{End}(\sigma)$ for non-commutative groups can be viewed as a generalization of Fourier transform.

7. Induction of representations

We are looking for a way of "lifting" representations of a subgroup $H < G$ to representations of G. In other words, we are looking for a "functor" $Ind_H^G : Rep(H) \to Rep(G)$.

Let us first find the trace (character) ψ of $\mathrm{Ind}_H^G(\pi)$. We have a natural map $Res_H^G : Z(\mathcal{A}(G)) \to Z(\mathcal{A}(H))$. On both algebras we have a natural non-degenerate bilinear form. Let us define $\mathrm{Ind}_H^G : Z(\mathcal{A}(H)) \to Z(\mathcal{A}(G))$ as the conjugate to Res_H^G w.r. to these forms. For any $g \in G$ let C_g denote the conjugacy class of g and δ_{C_g} denote the function which equals $|C_g|^{-1}$ on C_g and zero outside C_g. Then the functions of this form span $Z(\mathcal{A}(G))$. Denote by $\psi := \mathrm{Ind}_H^G(\chi)$, and let ψ^* denote the function on G given by $\psi^*(g) := \psi(g^{-1})$. Now, by definition

$$\psi(g) = |G|\langle C_g \, , \, \psi^* \rangle_G = |G|\langle \delta_{C_g}|_H, \chi_\pi^* \rangle = \frac{|G|}{|H||C_g|} \sum_{h \in C_g \cap H} \chi_\pi(h)$$

As we know, this defines $\mathrm{Ind}_H^G(\pi)$ uniquely (up to isomorphism). One only has to show existence now. However, before doing this let us check the meaning of induction by evaluating $\mathrm{Ind}_H^G(\chi_\pi)$ on another (generating) subset of $Z(\mathcal{A}(G))$ - the one formed by characters of representations.

$$\langle \tau, \mathrm{Ind}_H^G(\pi) \rangle = \langle \chi_\tau, \mathrm{Ind}_H^G(\chi_\pi) \rangle_G = \langle Res_H^G \chi_\tau, \chi_\pi \rangle_H = \langle Res_H^G \tau, \pi \rangle$$

This very important formula is called Frobenius reciprocity. First of all, it shows that $\mathrm{Ind}_H^G(\chi_\pi)$ is the character of a representation. It also defines induction uniquely and in fact could be guessed without considering characters since in means that $\mathrm{Ind}_H^G(\pi)$ is the "free representation of G generated by π". Similar definitions work for the free group, free module etc.

Let us now construct $\mathrm{Ind}_H^G(\pi)$. First let us consider several examples

Example 7.1. *(1) $H = \{e\}$, $\mathrm{Ind}_H^G(F) = F(G)$.*
(2) For any $H, \mathrm{Ind}_H^G(F) = F(G/H)$.
(3) For any character χ of H, $\mathrm{Ind}_H^G(\chi) = \{f \in F(G) : f(gh) = \chi(h)f(g)\}$.
(4) For any H-set X, the free G-set generated by X is the set of H-orbits in $G \times X$ under the action $h(g, x) := (gh^{-1}, hx)$.

Based on these we define, for any $(\pi, V) \in Rep(H)$,

$$\mathrm{Ind}_H^G(\pi) = \{f \in F(G, V) : f(gh) = \pi(h^{-1})f(g)\},$$

where $F(G, V)$ denotes all the functions from G to V with the usual action of G, i.e. $\mathrm{Ind}_H^G(\pi)(g)f(g') = f(g^{-1}g')$.

Moreover, this construction is functorial. This means that for $\pi_1, \pi_2 \in Rep(H)$ and $\phi \in \mathrm{Hom}_H(\pi_1, \pi_2)$ we define $\mathrm{Ind}_H^G(\phi) : \mathrm{Ind}_H^G(\pi_1) \to \mathrm{Ind}_H^G(\pi_2)$ by $\mathrm{Ind}_H^G(\phi)(f)(g) = \phi(f(g))$, and this preserves composition.

Lemma 7.2. *The above construction satisfies Frobenius reciprocity. More precisely, for any $\pi \in Rep(H)$ and $\tau \in Rep(G)$ there is a canonical isomorphism*

$$\mathrm{Hom}_G(\tau, Ind_H^G(\pi)) \simeq \mathrm{Hom}_H(\tau|_H, \pi)$$

Proof. To build the isomorphism let $\phi : \tau \to Ind_H^G(\pi)$. Then its image is given by $\psi(w) = (\phi(w))(e)$, where $w \in \tau$, and $e \in G$ is the identity element. The inverse morphism maps $\psi \in \mathrm{Hom}_H(\tau|_H, \pi)$ to $\phi \in \mathrm{Hom}_G(\tau, Ind_H^G(\pi))$ defined by $\phi(w)(g) := \psi(g^{-1}w)$. □

Exercise 7.3. *(1) For $H < G$ and $\pi_1, \pi_2 \in Rep(H)$,*

$$\mathrm{Ind}_H^G(\pi_1 \oplus \pi_2) = \mathrm{Ind}_H^G(\pi_1) \oplus \mathrm{Ind}_H^G(\pi_2).$$

(2) For $H_1 < H_2 < G$ and $\pi \in Rep(H)$,

$$\mathrm{Ind}_{H_2}^G \mathrm{Ind}_{H_1}^{H_2} \pi = \mathrm{Ind}_{H_1}^G \pi$$

Exercise 7.4. *Repeat Exercise 4.6 for a dodecahedron.*

Hint. Let G be the symmetry group of the dodecahedron, and H be the stabilizer of a face. Then the set of faces is G/H, and we consider the space of functions $F(G/H)$ as a representation of G. It is the induction of the trivial representation: $F(G/H) = \mathrm{Ind}_H^G 1$. As in the case of the cube, the averaging operator L is an intertwining operator. As in the cube again, one can show that $F(G/H)$ is a multiplicity free representation, and thus L diagonalizes - it acts by a scalar on each irreducible subrepresentation of $F(G/H)$. We want to find those scalars. By Frobenius reciprocity, every irreducible subrepresentation has an H-invariant vector. Compute L on such vectors. □

Induction can be best described using equivariant sheaves.

7.1. **Induction and equivariant sheaves.** In this section we will use two topological notions: vector bundles and equivariant sheaves. Since we consider only finite sets with discrete topology, in our case these notions become much simpler.

Intuitively, a sheaf is a continuous family of vector spaces, parameterized by points of a given topological space X. If we demand that all the spaces have the same dimension we will get a vector bundle. In our case, these are precisely the definitions, and we require the dimensions to be finite.

We will denote sheaves by Gothic letters, mainly $\mathcal{F}$. Let $\mathcal{F}$ be a sheaf over X. The vector space corresponding to $x \in X$ is called the **fiber** of $\mathcal{F}$ at x and denoted $\mathcal{F}_x$. The disjoint union of all fibers is called the **total space** of $\mathcal{F}$ and we denote it by $T(\mathcal{F})$. Note that we have a natural map $T(\mathcal{F}) \to X$, and that $T(\mathcal{F})$, together with the map $T(\mathcal{F}) \to X$ defines $\mathcal{F}$ uniquely.

A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ over the same space X is a collection of linear maps $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$, one for each $x \in X$.

For any (open) subset $U \subset X$, we define $\mathcal{F}(U) := \bigoplus_{x \in U} \mathcal{F}_x$. This space is called the space of sections of $\mathcal{F}$ on U since it is precisely the space of sections of $T(\mathcal{F}) \to X$ on U. The space $\mathcal{F}(X)$ is called the space of global sections and sometimes denoted $\Gamma(\mathcal{F})$.

Now, for a (continuous) map $\nu : X \to Y$ define $\nu_* : Sh(X) \to Sh(Y)$ and $\nu^* : Sh(Y) \to Sh(X)$ by

$$\nu_*(\mathcal{F})(U) := \mathcal{F}(\nu^{-1}(U)) \quad \text{and} (\nu^*(\mathcal{G}))_x := \mathcal{G}_{\nu(x)},$$

where $\mathcal{F} \in Sh(X)$ and $\mathcal{G} \in Sh(Y)$.

Exercise 7.5. *Let $\nu : X \to Y$ and let $\mathcal{F}_1, \mathcal{F}_2 \in Sh(X)$, $\mathcal{G}_1, \mathcal{G}_2 \in Sh(Y)$, $\phi : \mathcal{F}_1 \to \mathcal{F}_2$, $\psi : \mathcal{G}_1 \to \mathcal{G}_2$. Define natural maps $\nu_*(\phi) : \nu_*(\mathcal{F}_1) \to \nu_*(\mathcal{F}_2)$ and $\nu^*(\psi) : \nu^*(\mathcal{G}_1) \to \nu^*(\mathcal{G}_2)$.*

Definition 7.6. *Let X be a G-set and $\mathcal{F}$ be a sheaf over X. A G-equivariant structure on $\mathcal{F}$ is a G-set structure on the total space $T(\mathcal{F})$ such that the natural map $T(\mathcal{F}) \to X$ is a morphism of G-sets.*

Exercise 7.7. *The following structures on $\mathcal{F}$ are equivalent:*

(1) An equivariant structure
(2) For any $x \in X$ and $g \in G$ - a linear map $\pi(g)_x : \mathcal{F}_x \to \mathcal{F}_{gx}$ such that for $g_1, g_2 \in G$, $\pi(g_1 g_2)_x = \pi(g_1)_{g_2 x} \circ \pi(g_2)_x$.
(3) An isomorphism of sheaves $\alpha : a^*(\mathcal{F}) \approx \mathfrak{p}_2^*(\mathcal{F})$, where $p_2, a : G \times X \to X$ are the projection to the second coordinate and the action respectively, that satisfies the following condition:
(*) Consider the set $Z = G \times G \times X$ and two morphisms $q, b : Z \to X$, defined by $q(g, g', x) = x$ and $b(g, g', x) = gg'x$. The morphism α induces two morphisms of sheaves $\beta, \gamma : q^*(\mathcal{F}) \to b^*(\mathcal{F})$. The condition on α is that these two morphisms are equal.

Definition 7.8. *Let $\mathcal{F}, \mathcal{H} \in Sh_G(X)$. Then a morphism of equivariant sheaves $\mathcal{F} \to \mathcal{H}$ is a morphism of sheaves such that the corresponding map of total spaces $T(\mathcal{F}) \to T(\mathcal{H})$ is a morphism of G-sets.*

Exercise 7.9. *Give the definition of a morphism of equivariant sheaves in two other realizations of equivariant sheaves.*

We have the following obvious lemma.

Lemma 7.10. *Let $X = X_1 \coprod X_2$ be a disjoint union of G-sets. Then*
$$Sh_G(X) = Sh_G(X_1) \oplus Sh_G(X_2).$$

Definition 7.11. *For an equivariant sheaf $\mathcal{F}$ on X, define the action of G on the space of global sections $\mathcal{F}(X)$ by $(gf)(x) := g(f(g^{-1}x))$, where $f : X \to T\mathcal{F}$ is a global section.*

Corollary 7.12. *If $\mathcal{F} \in Sh_G(X)$ and $\mathcal{F}(X)$ is irreducible then either $\mathcal{F}(X_1) = 0$ or $\mathcal{F}(X_2) = 0$.*

Let us now study sheaves over a transitive G-set, G/H.

Lemma 7.13. *There is a natural equivalence $Sh_G(G/H) \cong Rep(H)$.*

Proof. Given a sheaf on G/H, we take its fiber at the coset H. To a representation (π, V) of H, we put in correspondence the vector bundle $\mathcal{I}nd(\pi)$ whose total space is the set of H-orbits in $G \times V$ under the action $h \cdot (g, v) := (gh^{-1}, \pi(h)v)$. The action of G on the total space is given by left multiplication.

Let us show that these two functors are indeed inverse to each other. One direction is easy: $\mathcal{I}nd(\pi)_H$ is identified with π by sending each equivalence class of pairs (h, v) to the unique representative of the form $(1, v)$. Checking the action of H: $h(1, v) = (h, v) \sim (1, \pi(h)v)$.

To the other direction, let $\mathcal{F}$ be a G-equivariant sheaf on G/H, and let $\pi := \mathcal{F}_H$. Then we have a natural map $\mathcal{I}nd(\pi) \to \mathcal{F}$ given on the total spaces by $(g, v) \mapsto gv_1$, where v_1 denotes the image of v in the total space $T\mathcal{F}$ (recall that $T\mathcal{F}$ is a union of fibers, and therefore $\pi = \mathcal{F}_H$ is naturally a subset). This map is well-defined, since $(gh, \pi(h^{-1})v) \mapsto$

$ghh^{-1}v = gv$. It is easy to see that the map commutes with the projections to G/H. It is left to check that the map is a bijection. It preserves fibers, is clearly a bijection on the fibers at H, therefore on the fibers at any other point, and therefore everywhere. □

To describe the fibers of $\mathcal{I}nd(\pi)$ at every point, choose a representative g_i for every coset and let $\mathcal{I}nd(\pi)_{g_iH}$ be the representation (π^{g_i}, V) of $g_iHg_i^{-1}$ given by $\pi^{g_i}(g_ihg_i^{-1}) = \pi(h)$. The map $\mathcal{I}nd(\pi)_H \to \mathcal{I}nd(\pi)_{g_iH}$ given by g_i is the identity map, and all other maps are compositions of the above 2 types.

Exercise 7.14. $\mathcal{I}nd(\pi)(G/H) = Ind_H^G(\pi)$.

Let x_i be a set of representatives of G-orbits on X and G_i be the stabilizers in G of x_i. Then the above discussion defines an equivalence $Sh_G(X) \simeq \bigoplus_i Rep(G_i)$.

8. Mackey theory

Let $N < G$ be a normal subgroup. Let $\pi \in Rep(G)$ and let $\pi|_N = \bigoplus_{\sigma \in Irr(N)} \pi_\sigma$ be the decomposition of $\pi|_N$ to isotypic components. For any representation (ρ, V) of N, and $g \in G$ we have a new representation (ρ^g, V) of N by $\rho^g(n)v := \rho(gng^{-1})v$. Note that $gng^{-1} \in N$. This defines an action of G on $\mathrm{Irr}(N)$.

The decomposition of $\pi|_N$ to isotypic components defines an equivalence

$$Rep(G) \simeq Sh_G^{spec}(Irr(N)),$$

where by $Sh_G^{spec}(Irr(N))$ we mean the sheaves on $Irr(N)$ such that the fiber at each point ρ is an isotypic representation of N of type ρ. Let σ_i be a set of representatives of orbits of G on $Irr(N)$ and S_i be the stabilizers in G of σ_i. Then $Rep(G) \simeq \bigoplus_i Rep^{spec}(S_i)$, where $Rep^{spec}(S_i)$ denotes the category of representations whose restrictions to N are isotypic of type σ_i. In particular, if $\sigma \in Irr(S_i)$ then $\mathrm{Ind}_{S_i}^G(\sigma) \in Irr(G)$, and any irreducible representation of G is obtained in this way.

Corollary 8.1. *Let $\pi \in Irr(G)$. Then either $\pi|_N$ is isotypic of type (ρ, V) and $\rho^g \approx \rho$ for all $g \in G$, or there exists a subgroup $N < H \subsetneq G$ and an irreducible representation τ of H such that $\pi = \mathrm{Ind}_H^G(\tau)$.*

Proof. By the discussion above, $\pi = \bigoplus_{i=1}^k \mathrm{Ind}_{S_i}^G \sigma_i$. Since π is irreducible, $k = 1$. If $S_1 = G$ then for $\rho = \sigma_1$ we have that $\pi|_N$ is isotypic of type (ρ, V) and $\rho^g \approx \rho$ for all $g \in G$. If $S_1 \neq G$ let $H := S_1$ and $\tau := \sigma_1$. Then $\pi = \mathrm{Ind}_H^G(\tau)$. □

Note that in the first case we get a projective representation of G on V, i.e. a group homomorphism $G \to GL(V)/scalars$. Indeed, for any $g \in G$, ρ is isomorphic to ρ^g. This isomorphism is a linear operator on V. Denote it by $\tau(g)$. To be a representation, $\tau(gh)$ should equal $\tau(g)\tau(h)$. We do not have this equality, but both are intertwining operators from ρ to ρ^{gh}. Thus, by the second Schur's lemma, their ratio is a scalar. Thus τ defines a projective representation.

Now suppose that N is commutative and $G = S \ltimes N$.

Exercise 8.2. *For any $\pi \in Irr(G)$, $\dim \pi \leq |S|$.*

Now, consider

$$P_2(\mathbb{F}_q) := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q \right\}$$

Example 8.3. *Note that $P_2 = \mathbb{F}_q^\times \ltimes \mathbb{F}_q$. There are 2 orbits of $\mathbb{F}_q^\times$ on the dual group $\widehat{\mathbb{F}_q}$ (consisting of characters): the zero and the non-zero orbit. The stabilizers are $\mathbb{F}_q^\times$ and the trivial group respectively. Fix a non-trivial character ψ of $\mathbb{F}_q$. Then there are q irreducible representations of P_2: $\mathrm{Ind}_{\mathbb{F}_q}^{P_2}(\psi)$ and $q-1$ characters of $\mathbb{F}_q^\times$, continued trivially to P_2.*

Let us now extend this example to $P_n = \mathrm{GL}_{n-1}(\mathbb{F}_q) \ltimes \mathbb{F}_q^{n-1}$. All the non-trivial characters of $\mathbb{F}_q^{n-1}$ are conjugate under $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ to the character ψ_{n-1} defined by $\psi_{n-1}(a_1, \ldots, a_{n-1}) = \psi(a_{n-1})$. Furthermore, the stabilizer of ψ_{n-1} is P_{n-1}.

This enables to reduce the classification of the irreducible representations of P_n to the classifications of all irreducible representations of $\mathrm{GL}_k(\mathbb{F}_q)$ for all $k < n$ in the following way. Let $\pi \in \mathrm{Irr}(P_n)$. If $\pi|_{\mathbb{F}_q^{n-1}}$ is isotypic then it is a multiple of the trivial character, since the trivial character is the only invariant character. If this is indeed the case, then $\pi|_{\mathrm{GL}_{n-1}(\mathbb{F}_q)}$ is irreducible.

If $\pi|_{\mathbb{F}_q^{n-1}}$ is not isotypic then it includes all non-trivial characters, because they are all conjugate under $\mathrm{GL}_{n-1}(\mathbb{F}_q)$. Thus $\pi \cong \mathrm{Ind}_{Q_n}^{P_n}(\rho \otimes \psi_{n-1})$, where $Q_n = P_{n-1} \ltimes \mathbb{F}_q^{n-1}$, and $\rho \in \mathrm{Irr}(P_{n-1})$. To classify ρ we continue by induction: restrict it to $\mathbb{F}_q^{n-2}$ and so on. Eventually we “drop” to the GL_k side for some $k < n$.

To summarize: for any $\pi \in \mathrm{Irr}(P_n)$ there exist $0 \leq k < n$ and $\rho \in \mathrm{Irr}(\mathrm{GL}_k(\mathbb{F}_q))$ s.t.

$$\pi \cong \mathrm{Ind}_{R_n^k}^{P_n}(\rho \otimes 1_k \otimes \psi_{k+1} \otimes \psi_{k+2} \otimes \cdots \otimes \psi_{n-1}),$$

where

$$R_n^k := \mathrm{GL}_k(\mathbb{F}_q) \ltimes (\mathbb{F}_q^k \oplus \mathbb{F}_q^{k+1} \oplus \mathbb{F}_q^{k+2} \oplus \cdots \oplus \mathbb{F}_q^{n-1})$$

For example,

$$R_5^2(\mathbb{F}_q) = \left\{ \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 1 & \underline{*} & * \\ 0 & 0 & 0 & 1 & \underline{*} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

Now, let G be a general finite group and $K, H < G$ be two subgroups and $\pi \in Rep(H)$. Let us study $(\mathrm{Ind}_H^G(\pi))|_K$ using equivariant sheaves. We know that $\mathrm{Ind}_H^G(\pi)$ is the space of global sections of the equivariant sheaf $\mathcal{I}nd_H^G(\pi)$ on G/H. Clearly, the orbits of K in G/H are the double-cosets $K\backslash G/H$. Note that

$$\mathcal{I}nd_H^G(\pi)(KgH) = \bigoplus_{k \in K/(K \cap gHg^{-1})} \pi^{kg} = \mathrm{Ind}_{K \cap gHg^{-1}}^K(\pi^g)$$

Thus,

Theorem 8.4.

$$\mathrm{Ind}_H^G(\pi)|_K = \bigoplus_{KgH \in K\backslash G/H} \mathrm{Ind}_{K \cap gHg^{-1}}^K(\pi^g)$$

Corollary 8.5. *(i) $\langle \mathrm{Ind}_H^G(\pi), \mathrm{Ind}_K^G(\tau) \rangle_G = \sum_{KgH \in K\backslash G/H} \langle \pi^g, \tau \rangle_{K \cap H^g}$*

(ii) $\mathrm{Ind}_H^G(\pi)$ is irreducible if and only if π is irreducible and $\langle \pi, \pi^g \rangle_{H \cap H^g} = 0$ for any $g \notin H$.

Proof. (i)

$$\langle \mathrm{Ind}_H^G(\pi), \mathrm{Ind}_K^G(\tau)\rangle_G = \langle \mathrm{Ind}_H^G(\pi), \tau\rangle_K = \sum_{KgH\in K\backslash G/H} \langle \mathrm{Ind}_{K\cap H^g}^K(\pi^g), \tau\rangle_K = \sum_{KgH\in K\backslash G/H} \langle \pi^g, \tau\rangle_{K\cap H^g}$$

(ii) $\mathrm{Ind}_H^G(\pi)$ is irreducible if and only if $\langle \mathrm{Ind}_H^G(\pi), \mathrm{Ind}_H^G(\pi)\rangle = 1$. By part (i),

$$\langle \mathrm{Ind}_H^G(\pi), \mathrm{Ind}_H^G(\pi)\rangle_G = \sum_{HgH\in H\backslash G/H} \langle \pi^g, \pi\rangle_{H\cap H^g}$$

Now, for $g = 1$, $\langle \pi^g, \pi\rangle_{H\cap H^g} = \langle \pi, \pi\rangle_H \geq 1$. Thus, $\mathrm{Ind}_H^G(\pi)$ is irreducible if and only if $\langle \pi, \pi\rangle_H = 1$ and $\langle \pi, \pi^g\rangle_{H\cap H^g} = 0$ for any $g \notin H$. □

9. Monomial representations, Heisenberg group, Weil representation

We have seen in the last lecture that induction enables to construct many irreducible representations. Today we will see an extreme case of that: any irreducible representation of a nilpotent group is induced from a character.

We will use a lemma from last time:

Lemma 9.1. *Let $N \lhd G$ be a normal subgroup. Let $\pi \in Irr(G)$. Then either $\pi|_N$ is isotypic of some type (ρ, V) and $\rho^g \approx \rho$ for all $g \in G$, or there exists a subgroup $N < H \subsetneq G$ and an irreducible representation τ of H such that $\pi = \mathrm{Ind}_H^G(\tau)$.*

Note that in the first case we get a projective representation of G on V, i.e. a group homomorphism $G \to GL(V)/scalars$.

Definition 9.2. *A representation induced from a (1-dimensional) character of a subgroup is called monomial.*

Definition 9.3. *Let us call a group G c-solvable(which means cyclicly solvable) if there exists a sequence of normal subgroups $N_0 < N_1 < ... < N_k = G$ starting with the trivial subgroup N_0 such that each quotient group N_i/N_{i-1} is cyclic.*

Exercise 9.4. *Show that any subgroup and quotient group of a c-solvable group is c-solvable. Show that any finite nilpotent group is c-solvable.*

Theorem 9.5. *Let G be a c-solvable finite group. Then any irreducible representation π of G is monomial.*

Proof. We prove the theorem by induction on the order of G. If the group is commutative the theorem is clear.

Suppose that the group is not commutative. We may also suppose that the representation π is faithful, i.e. no group element acts trivially. Now, let $Z < G$ denote the center. Choose a normal cyclic subgroup $C < G/Z$ and lift it to a normal commutative subgroup $N < G$. Since N is not central, there exist $a \in N$ and $b \in G$ such that $a \neq bab^{-1}$, thus $\pi(a) \neq \pi(bab^{-1})$.

By Lemma 9.1, either $\pi|_N$ is isotypic and isomorphic to $\pi^b|_N$, or π is induced from some proper subgroup of G. Since N is commutative, if $\pi|_N$ is isotypic then all elements of N act on π by scalars. But $\pi(a) \neq \pi(bab^{-1})$ and thus $\pi|_N$ is not isomorphic to $\pi^b|_N$. Thus π is induced from some subgroup. By the induction hypotheses the representation of the subgroup is monomial, and by transitivity of induction π is monomial. □

Exercise 9.6. *Suppose we know that a group G has a commutative normal subgroup N such that the group G/N is c-solvable. Show that any irreducible representation σ of G is monomial.*

Definition 9.7. *The Heisenberg group is the group of upper uni-triangular 3 by 3 matrices (over some field which we will take to be $\mathbb{F}_q$).*

$$\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & x' & z'\\ 0 & 1 & y'\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & x+x' & z+z'+xy'\\ 0 & 1 & y+y'\\ 0 & 0 & 1\end{pmatrix}$$

Let us classify all irreps of H. First of all, on every irrep the center $Z \cong \mathbb{F}_q$ acts by some character. If the character is trivial, we get an irreducible representation of V - there are q^2 such representations and they are all 1-dimensional. Now, suppose the central character is $\chi \neq 1$.

Theorem 9.8. *There exists a unique irreducible representation ρ_χ of H with central character χ, and it has dimension q.*

Proof. Define a normal commutative subgroup $D = \{(x, y, z) \in H : x = 0\}$. Extend χ trivially to D and define $\rho_\chi := Ind_D^H(\chi)$. The irreducibility and uniqueness follow from the above Mackey analysis.

Indeed, for irreducibility we use Corollary 8.5. Let $g = (x, y, z) \in G$ with $x \neq 0$. Then, for any $(0, y', z') \in D$, $\chi^g((0, y', z') = \chi(z')\chi(1/2xy')$. Since $x \neq 0$ and $\chi \neq 1$, there exists y' such that $\chi(1/2xy') \neq 1$ and thus $\langle\chi, \chi^g\rangle_D = 0$.

To show uniqueness, let $\sigma \in Irr(H)$ and consider $\sigma|_D$. By Lemma 9.1, either $\sigma|_D$ is isotypic, or σ is induced from some proper subgroup which includes D. In the first case, D acts on σ by scalars, and $(0, y', z')$ and $(0, y', z' + 1/2xy')$ act by the same scalar for any $x, y', z' \in \mathbb{F}_q$. However, this implies $\chi(1/2xy') = 1$ for all $x, y' \in \mathbb{F}_q$ which contradicts $\chi \neq 1$. Thus σ is induced from a representation τ of some proper subgroup which includes D. If this subgroup is bigger than D we apply the same argument to show that τ is induced from a smaller subgroup. Eventually, we get that $\sigma = \mathrm{Ind}_D^H \chi'$ where $\chi'|_Z = \chi$. Since H conjugates any such character χ' to χ, we obtain $\sigma \simeq \rho_\chi$.

Another option is to deduce irreducibility directly from the construction of induction, and uniqueness will follow from the dimension count (sum of squares of dimensions).

Explicit construction of ρ_χ: (x, y, z) acts on $F(\mathbb{F}_q)$ by

$$(x, y, z)f(x') = \chi(z)\chi(x'y)f(x' - x).$$

$\square$

Here is another description of the Heisenberg group. The center of H is $\mathbb{F}_q$ (the corner of the matrix). The other two entries (x, y) form a 2-dimensional vector space V over $\mathbb{F}_q$, and on this vector space we define a form $\omega((x_1, y_1), (x_2, y_2)) := x_1y_2 - x_2y_1$. It is anti-symmetric and non-degenerate. Now, H is isomorphic to $\{v, z : v \in V, z \in \mathbb{F}_q\}$ with group law given by

$$(v, z)(v', z') = (v + v', z + z' + \frac{1}{2}\omega(v, v'))$$

The isomorphism is given by

$$(x, y, z) \mapsto ((x, y), z + 2^{-1/2}xy).$$

Now, note that the group $\mathrm{SL}_2(\mathbb{F}_q)$ of 2 by 2 matrices with determinant 1 acts on H by automorphisms through its action on $V = \mathbb{F}_q^2$ and trivial action on Z. Thus, it maps ρ_χ to itself. This defines a projective representation of $SL_2(\mathbb{F}_q)$ on $F(\mathbb{F}_q)$. In fact, this representation can be lifted to an honest representation in the following way:

$$\rho_\chi \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} f(x) = \left(\frac{a}{p}\right) f(a^{-1}x); \qquad \rho_\chi \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} f(x) = \chi\left(\frac{1}{2}bx^2\right) f(x);$$
$$\rho_\chi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f = -\frac{i^{(q-1)/2}}{\sqrt{q}} FT(f),$$

where $q = p^n$, $p \neq 2$, and $\left(\frac{a}{p}\right)$ denotes the Legendre symbol, and FT denotes the Fourier transform.

Theorem 9.8 generalizes to representations of higher Heisenberg groups

$$H_n := (\mathbb{F}_q^n \times \mathbb{F}_q^n) \ltimes \mathbb{F}_q.$$

The Weil representation in this case is a projective representation representation of the symplectic group $Sp_{2n}(\mathbb{F}_q)$.

An analogous theory holds over the reals (instead of $\mathbb{F}_q$), but the Weil representation stays a projective representation even for $n = 2$ and does not lift to an "honest" representation. The analog of Theorem 9.8 for $H_n(\mathbb{R})$ is called the Stone-von-Neumann theorem.

10. Gelfand Pairs with applications to representations of symmetric groups

Let $H < G$ be finite groups.

Definition 10.1. *(G, H) is called a Gelfand pair if for every $\pi \in \mathrm{Irr}(G)$, $\dim \pi^H \leq 1$.*

Exercise 10.2. *(i) $(G \times G, \Delta G)$ is a Gelfand pair.*
(ii) If G is commutative then (G, H) is a Gelfand pair for any subgroup $H \subset G$.
(iii) If $H \subset G$ is normal then (G, H) is a Gelfand pair if and only if G/H is commutative.
(iv) (S_3, S_2) is a Gelfand pair.

Lemma 10.3. *(G, H) is a Gelfand pair if and only if $F(G/H)$ is a multiplicity free representation of G.*

Proof. Note that $F(G/H) = \mathrm{Ind}_H^G(F)$ and that $\mathrm{Hom}_H(F, \pi) = \pi^H$. Using the Frobenius reciprocity we have

$$\dim \pi^H = \langle F, \pi \rangle_H = \langle \mathrm{Ind}_H^G(F), \pi \rangle_G$$

□

Corollary 10.4. *(S_{n+1}, S_n) is a Gelfand pair.*

Definition 10.5. *An F-algebra (or just an algebra) is a ring that includes F. Its dimension is its dimension as an F-vector space.*

Theorem 10.6. *(G, H) is a Gelfand pair if and only if the convolution algebra $\mathcal{A}(G)^{H \times H}$ of functions on G that are constant on H double cosets is commutative.*

For the proof we will need the following facts on finite-dimensinoal algebras.

Definition 10.7. *A module M over an algebra A is a vector space over F together with a function $A \times M \to M$ called product, that extends the product $F \times M \to M$ and satisfies the usual associativity and distributivity axioms: $a(bm) = (ab)m$, $a(m+n) = am + an$ and $(a+b)m = am + bm$.*

Example 10.8. *An F-module is the same as a vector space over F. An $\mathcal{A}(G)$-module is the same as a representation of G.*

Definition 10.9. *An A-module is called simple if it has no non-trivial submodules. The collection of all simple A-modules is denoted* $\mathrm{Irr}(A)$.

Definition 10.10. *An element $a \in A$ is called* strongly nilpotent *if there exists $n > 0$ such that for any $b_1, \dots b_n \in A$, we have $ab_1ab_2 \dots ab_n = 0$.*

Example 10.11. *$\mathcal{A}(G)$ has no strong nilpotents, and neither does any its subalgebra.*

Theorem 10.12. *Let A be a finite-dimensional F-algebra without (non-zero) strong nilpotents. Then*

$$A \cong \oplus_{M \in \mathrm{Irr}(A)} \mathrm{End}_F(M).$$

We proved this theorem in the case when $A = \mathcal{A}(G)$. The same proof works in general.

Corollary 10.13. *Let A be a finite-dimensional F-algebra without (non-zero) strong nilpotents. Then*

(i) A is commutative if and only if all simple A-modules are 1-dimensional.
(ii) $\dim A = \sum_{M \in \mathrm{Irr}(A)} (\dim M)^2$

The assumption on strong nilpotents is necessary, as shown in the following example.

Example 10.14. *Let A be the commutative algebra consisting of all matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. Then A has only one simple module M. This module is one-dimensional, and the nilpotent element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acts on it by zero.*

Lemma 10.15. *Let G be a finite group and $H \subset G$ be a subgroup. Let $A := \mathcal{A}(G)^{H \times H}$ as an algebra with convolution as a product. Then for every $\pi \in \mathrm{Irr}(G)$, the space π^H is either zero, or a simple A-module.*

Proof. Assume that $\pi^H \neq 0$ and let $V \subset \pi^H$ be a non-zero A-submodule. We have to show that $V = \pi^H$. Define $e_H \in A$ by

$$e_H(g) := \begin{cases} |H|^{-1} & g \in H \\ 0 & g \notin H \end{cases}$$

Then e_H acts on π by averaging over H. Let $v \neq 0 \in V$ and $w \in \pi^H$. Then for some $f \in \mathcal{A}(G)$ we have $\pi(f)v = w$. Furthermore, since v and w are H-invariant, we have $e_H f e_H v = e_H w = w$. Now, $e_H f e_H \in A$, and thus $w \in V$ and $V = \pi^H$. $\square$

Proof of theorem 10.6. Let $A := \mathcal{A}(G)^{H\times H}$. By Corollary 10.13(i), A is commutative if and only if all simple A-modules are one-dimensional. By Lemma 10.15, π^H is a simple A-module for any $\pi \in \mathrm{Irr}(G)$. Thus it is enough to show that all simple A-modules are of this form. By Corollary 10.13(ii), it is enough to show that

$$\sum_{\pi\in \mathrm{Irr}(G)} (\dim \pi^H)^2 = \dim A.$$

For that, note that by Frobenius reciprocity

$$\dim A = \dim \mathcal{A}(G)^{H\times H} = \langle F, F(G/H)\rangle_H = \langle F(G/H), F(G/H)\rangle_G$$

and

$$F(G/H) = \bigoplus_{\pi\in Irr(G)} (\dim \pi^H)\pi$$

and thus

$$\sum_{\pi\in Irr(G)} (\dim \pi^H)^2 = \langle F(G/H), F(G/H)\rangle_G = \dim \mathcal{A}(G)^{H\times H}.$$

□

Remark 10.16. *This topic belongs to "relative representation theory", namely harmonic analysis on G/H (while the usual representation theory is harmonic analysis on G). We see that Gelfand property replaces Schur's lemma for relative representation theory. This explains why it is important.*

That theorem is great, since this reduces a statement on representations that we maybe do not know yet to an explicit statement on commutativity of algebras, that we can check by a direct computation. However, Gelfand and (independently) Selberg invented a trick that allows to avoid even that computation.

Lemma 10.17. *Suppose we have a bijection $\sigma : G \to G$ s.t. $\sigma(gh) = \sigma(h)\sigma(g)$. Suppose also that σ preserves all H double-cosets. Then $\mathcal{A}(G)^{H\times H}$ is commutative and thus (G, H) is a Gelfand pair.*

Proof. σ acts as identity on $\mathcal{A}(G)^{H\times H}$, but changes order of multiplication. Thus, this algebra is commutative. □

This lemma is obvious but very useful.

Exercise 10.18. *Prove $(S_{n+2}, S_n \times S_2)$ is a Gelfand pair.*

One can formulate a stronger property.

Definition 10.19. *(G, H) is called a strong Gelfand pair if for every $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(H)$, $\langle \pi|_H, \tau\rangle \leq 1$.*

Theorem 10.20. *The following are equivalent:*

(1) (G, H) is a strong Gelfand pair
(2) $(G \times H, \Delta H)$ is a Gelfand pair.
(3) For any $\tau \in \mathrm{Irr}(H)$, $\mathrm{Ind}_H^G(\tau)$ is a multiplicity free representation of G.
(4) The convolution algebra $\mathcal{A}(G)^{Ad(H)}$ of functions on G that are constant on H-conjugacy classes is commutative

Proof. (1) $\Leftrightarrow$ (3) by Frobenius reciprocity. For (1) $\Leftrightarrow$ (2) note that every irreducible representation of $G \times H$ has the form $\pi^* \otimes \tau \cong \mathrm{Hom}_F(\pi, \tau)$ where $\pi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Irr}(H)$. For (1) $\Leftrightarrow$ (4) note that

$$\mathcal{A}(G \times H)^{\Delta H \times \Delta H} \cong \mathcal{A}(G)^{Ad(H)}.$$

Indeed, this isomorphism is restriction to $G \times \{1\}$. To see the invariance, let f' be the restriction of f. Then $f'(hgh^{-1}) = f(hgh^{-1}, 1) = f(hg, h) = f(g, 1)$. The equivalence (1) $\Leftrightarrow$ (4) follows now from Theorem 10.6. $\square$

This theorem gives the following version of Gelfand - Selberg trick for strong Gelfand pairs:

Lemma 10.21. *Suppose we have a bijection $\sigma : G \to G$ that satisfies $\sigma(gh) = \sigma(h)\sigma(g)$ and preserves all H-conjugacy classes, i.e. that $\forall g \in G \, \exists h \in H$ s.t. $\sigma(g) = hgh^{-1}$. Then $\mathcal{A}(G)^{Ad(H)}$ is commutative and thus (G, H) is a strong Gelfand pair.*

Corollary 10.22. *(S_{n+1}, S_n) is a strong Gelfand pair.*

Proof. Let $\sigma(g) := g^{-1}$. By the lemma, it is enough to show that every permutation is conjugate under S_n to its inverse. Conjugation under S_n is equivalent to reenumeration of the elements $1, \dots n$. To invert the permutation, we decompose it into a product of cycles, and re-enumerate within each cycle to go backwards. One of the cycles will include the number $n + 1$ which we are not allowed to touch, but we can fix it and re-enumerate the rest. $\square$

Now we see that (S_n, S_{n-1}) is actually a strong Gelfand pair. We use this in the following way. Take $\pi \in \mathrm{Irr}(S_n)$. Then $\pi|_{S_{n-1}}$ is multiplicity free and thus has a canonical decomposition to a direct sum of irreducible subrepresentations. Take each of those subrepresentations, restrict it to S_{n-2} and so on. At the end, we get a decomposition of π to a direct sum of lines, i.e. a canonical bases up to multiplication by constants. It is very nice to have a canonical basis. We will use this basis to compute the dimensions of irreducible representations.

Recall that the irreducible representations of S_n are classified by partitions of n. For every partition $\lambda = (n_1, ..., n_k)$ we defined a set $X_\lambda = S_n/(S_{n_1} \times ... \times S_{n_k})$ and representations $T_\lambda := F(X_\lambda) = \mathrm{Ind}_{(S_{n_1} \times ... \times S_{n_k})}^{S_n} F$ and $T'_\lambda = \mathrm{sign}\, T_\lambda$. We showed that $\langle T_\lambda, T'_{\lambda^*} \rangle = 1$ and defined U_λ to be the unique irreducible representation that they have in common.

Theorem 10.23. *Let λ be a partition of n and ν be a partition of $n - 1$. Then $\langle U_\lambda|_{S_{n-1}}, U_\nu \rangle = 1$ if ν can be obtained from λ by decreasing one part by one, and is zero otherwise.*

To prove this theorem, let us do some combinatorics. Recall the partial ordering on partitions given by

$$\lambda \leq \mu \iff \sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i \quad \forall 1 \leq j \leq n.$$

Now, from a partition $\nu = (n_1, ..., n_k)$ of $n - 1$ we construct two partitions of n:

$$\nu^l := (n_1, ..., n_k, 1) \text{ and } \nu^r := (n_1 + 1, ..., n_k).$$

Note that ν can be obtained from λ by decreasing one part by one if and only if $\nu^l \leq \lambda \leq \nu^r$. Note also that $(\nu^r)^* = (\nu^*)^l$.

Lemma 10.24.
$$\mathrm{Ind}_{S_{n-1}}^{S_n}(T_\nu) = T_{\nu^l} \text{ and } \mathrm{Ind}_{S_{n-1}}^{S_n}(T'_{\nu^*}) = T'_{(\nu^r)^*}$$

Proof. $T_\nu = \mathrm{Ind}_{(S_{n_1}\times \ldots \times S_{n_k})}^{S_{n-1}} F$ and thus $\mathrm{Ind}_{S_{n-1}}^{S_n}(T_\nu) = Ind_{(S_{n_1}\times \ldots \times S_{n_k}\times S_1)}^{S_n} F = T_{\nu^l}$. The second statement follows from this since $(\nu^r)^* = (\nu^*)^l$. □

The proof of the theorem is based on the following difficult combinatorial exercise, which generalizes Theorem 5.10.

Exercise 10.25.
$$\langle T_\alpha, T'_{\beta^*}\rangle = \#\{\lambda : \ \alpha \leq \lambda \leq \beta\}$$
In particular, if $\langle T_\alpha, T'_{\beta^*}\rangle > 0$ then $\alpha \leq \beta$.

Proof of Theorem 10.23. Let λ be a partition of n and ν be a partition of $n-1$. First, suppose that $\langle U_\lambda|_{S_{n-1}}, U_\nu\rangle \neq 0$, and thus is 1 since (S_n, S_{n-1}) is a strong Gelfand pair. Then $\langle U_\lambda, Ind_{S_{n-1}}^{S_n} U_\nu\rangle \neq 0$. By Lemma 10.24 this implies $U_\lambda \subset T_{\nu^l}$ and $U_\lambda \subset T'_{(\nu^r)^*}$. By Exercise 10.25, this implies that $\nu^l \leq \lambda \leq \nu^r$.

To prove the implication in the other direction, note that Exercise 10.25 is the counting argument that shows that if $\nu^l \leq \lambda \leq \nu^r$ then $U_\lambda \subset T_{\nu^l}$ and $U_\lambda \subset T'_{(\nu^r)^*}$ and thus $\langle U_\lambda|_{S_{n-1}}, U_\nu\rangle \neq 0$. □

Corollary 10.26. $\dim U_\lambda =$ *the number of ways to "erase" the "boxes" in λ one by one so that in each step we have a (non-increasing) partition.*

Note that this number also equals the number of "special" Young diagrams, i.e. the number of ways to write the numbers $1, ..., n$ in the rows of λ such that each row and column will have decreasing order. This number happens to be
$$\frac{n!\prod_{i<j}(l_i - l_j)}{l_1!...l_k!}, \text{ where } l_i = n_i + k - i,\ i = 1, ..., k.$$

11. Brauer Induction Theorem

Fix a finite group G.

Definition 11.1. *Let $C(G) \subset F(G)$ denote the subalgebra of conjugation-invariant functions, and $R(G) \subset C(G)$ denote the subring generated by characters of representations.*

For a subgroup $E \subset G$ denote by $\mathrm{Ind}_E^G : C(E) \to C(G)$ the linear map adjoint to restriction $Res_E^G : C(G) \to C(E)$ (see §7).

Note that for $\tau \in Rep(E)$ we have $\mathrm{Ind}_E^G(\chi_\tau) = \chi_{\mathrm{Ind}_E^G \tau}$. Note also that $\chi_{\pi\oplus\tau} = \chi_\pi + \chi_\tau$ and $\chi_{\pi\otimes\tau} = \chi_\pi \cdot \chi_\tau$. Thus one can view $R(G)$ as the ring generated by the semi-ring of all representations of G.

Exercise 11.2.
1. *$R(G)$ is generated (over $\mathbb{Z}$) by characters of irreducible representations.*
2. *For any $f \in R(G)$, there exists representations π and τ such that $f = \chi_\pi - \chi_\tau$.*

Definition 11.3. *Let p be a prime number. A finite group E is called p-elementary if $E = C_m \times S$, where C_m is a cyclic group of order m prime to p, and S is a p-group. E is called elementary if it is p-elementary for some p.*

Our goal in this section is to prove

Theorem 11.4 (Brauer Induction Theorem)**.** *The (additive) group $R(G)$ is spanned by functions of the form $\operatorname{Ind}_E^G(\chi)$, where $E \subset G$ is an elementary subgroup and χ is a one-dimensional representation of E.*

We will now make several reductions. First of all, define

Exercise 11.5. *Demonstrate the Brower Induction Theorem for S_3. Namely, show how the character of each of the irreducible representations from the classification in section 3 can be expressed as an integer linear combination of inductions of characters of cyclic subgroups (in this case all elementary subgroups are cyclic).*

$$I(G) := \sum_{\text{elementary } E} \operatorname{Ind}_E^G(R(E)) \subset R(G).$$

Lemma 11.6. *The subset $I(G)$ is an ideal and if $1 \in I(G)$ then Theorem 11.4 holds.*

Proof. To see that $I(G)$ is an ideal let $\pi \in Rep(G)$ and $\sigma, \rho \in Rep(E)$. Then

$$\operatorname{Ind}_E^G(\sigma \oplus \rho) = \operatorname{Ind}_E^G(\sigma) \oplus \operatorname{Ind}_E^G(\rho) \text{ and } \pi \otimes \operatorname{Ind}_E^G(\sigma) = \operatorname{Ind}_E^G(\pi|_E \otimes \sigma).$$

Now, if $1 \in I(G)$ then $I(G) = R(G)$. On the other hand, every elementary E is nilpotent, thus (by Theorem 9.5), every representation of E is induced from a character of some subgroup $E' \subset E$. Thus $R(G)$ is spanned by functions of the form $\operatorname{Ind}_{E'}^G \chi$. $\square$

Definition 11.7. *A character system Q is a correspondence which assigns to every finite group H a subring $Q(H)$ of the algebra $C(H)$ such that for any pair $H < H'$ we have*

$$\operatorname{Ind}_H^{H'}(Q(H)) \subset Q(H'), \quad Res_H^{H'}(Q(H')) \subset Q(H).$$

Example 11.8. *(i) $Q(H) = R(H)$.*
(ii) $Q(H) = C(H)$.
(iii) $Q(H) = C_{\mathbb{Z}}(H)$, the subring of integer-valued functions.

Notation 11.9. *Let n be the order of G, and $\mu_n \subset F$ be the group of n-th roots of 1. Let Λ denote the subring of F generated by μ_n. Define a character system R_Λ by*

$$R_\Lambda(H) := \Lambda \cdot R(H) \subset C(H).$$

Denote also

$$I_\Lambda(G) := \sum_{\textit{elementary } E} \operatorname{Ind}_E^G(R_\Lambda(E)) \subset R_\Lambda(G).$$

Lemma 11.10. *If $1 \in I_\Lambda(G)$ then Theorem 11.4 holds.*

For the proof we will need the following exercise.

Exercise 11.11. *There exists a homomorphism of groups $\nu : \Lambda \to \mathbb{Z}$ with $\nu(1) = 1$.*

Proof of Lemma 11.10. Let ν be as in the exercise. Notice that for any group H there exists a unique morphism of groups $\nu_H : R_\Lambda(H) \to R(H)$ such that $\nu(\lambda r) = \nu(\lambda)r$, $\forall \lambda \in \Lambda$, $r \in R(H)$. This is true since $R(H)$ has a basis $\rho_1, \ldots, \rho_r$ of irreps, which stays a basis in $C(H)$. Clearly the system of morphisms ν_H is compatible with restriction and induction. In particular, $\nu(I_\Lambda(G)) \subset I(G)$. Thus, if $1 \in I_\Lambda(G)$ then $1 \in I(G)$ and Theorem 11.4 holds by Lemma 11.6. □

Consider the character system $Q(H) = R_\Lambda(H) \cap C_\mathbb{Z}(H)$ and define

$$J := \sum_{\text{elementary } E} \mathrm{Ind}_E^G(Q(E)) \subset I_\Lambda(G).$$

By Lemma 11.10 it is enough to show that $1 \in J$. To prove this we will use the following exercise.

Exercise 11.12. *Let $L \simeq \mathbb{Z}^r$ be a lattice, and $A < B < L$ be subgroups. Suppose that $A + p^N L = B + p^N L$ for all primes p and all positive integers N. Then $A = B$.*

Lemma 11.13. *Suppose that for every prime p there exists a function $f \in J$ such that for every $g \in G$, $f(g)$ is prime to p. Then Theorem 11.4 holds.*

Proof. Since $J \subset I_\Lambda(G)$, Lemma 11.10 implies that if $1 \in J$ then Theorem 11.4 holds. Let $A := J$, $L := C_\mathbb{Z}(G)$ and B be the subgroup of L generated by A and 1. We have to show that $A = B$. Fix a prime number p and a positive integer N. Fix a function $f \in J$ such that for every $g \in G$, $f(g)$ is prime to p. Then $p|(f(g)^{p-1} - 1)$, and by induction $p^N|(f^{p^{N-1}(p-1)} - 1)$. Thus $1 \in A + p^N L$ for every N and p, thus $A = B$ and $1 \in J$. □

From now on we fix a prime number p. To construct f as in Lemma 11.13 we will need the following definition and (difficult) exercise.

Definition 11.14. *An element $g \in G$ is called p-regular if $ord(g)$ is prime to p and p-singular if $ord(g)$ is a power of p.*

Exercise 11.15 (Jordan decomposition)**.** *Every element of G can be uniquely written as $g = g_r g_s = g_s g_r$, where g_r is p-regular and g_s is p-singular.*

Note that the uniqueness of Jordan decomposition implies that the maps $g \mapsto g_r$ and $g \mapsto g_s$ are compatible with morphisms of groups. In particular, they map conjugacy classes into conjugacy classes.

Lemma 11.16. *Suppose that for any p-regular element $a \in G$ there exists a function $f_a \in J$ such that for any $x \in G$ with x_r conjugate to a, $f_a(x)$ is prime to p, and for any $x \in G$ with x_r not conjugate to a, $f_a(x)$ is 0. Then there exists a function $f \in J$ such that for every $g \in G$, $f(g)$ is prime to p.*

Proof. Take f to be the sum of the functions f_a, when a runs over a system of representatives of p-regular conjugacy classes. □

Now fix a p-regular $a \in G$, set $m := ord(a)$ and let D be the cyclic subgroup generated by a. Denote by $Z(a)$ the centralizer of a, fix a p-Sylov subgroup S of $Z(a)$ and set $E = D \times S \subset Z(a)$. It is easy to see that E is an elementary subgroup and the projection $pr : E \to D$ coincides with the map $x \mapsto x_r$. Define a function $\varphi \in C(E)$ by

$$\varphi(x) = 0 \text{ if } pr(x) \neq a \text{ and } \varphi(x) = m \text{ if } pr(x) = a.$$

Lemma 11.17. *The function φ lies in $Q(E)$.*

Proof. First of all, φ takes integer values. Also, we can write it in the form

$$\varphi = m \sum_{\chi} \chi(a^{-1})\chi',$$

where the sum is taken over all characters χ of the group D and χ' is the character of E defined by $\chi' = \chi(pr(x))$. Since the coefficients $\chi(a^{-1})$ lie in Λ we see that $\varphi \in R_\Lambda(E)$, and thus $\varphi \in Q(E)$. □

Proposition 11.18. *The induction $f_a := \mathrm{Ind}_E^G(\varphi)$ satisfies the conditions of Lemma 11.16.*

Theorem 11.4 follows now from Lemma 11.17, Proposition 11.18, Lemma 11.16 and Lemma 11.13.
For the proof of Proposition 11.18 we will need one more exercise.

Exercise 11.19. *Let Y be a finite set, t be a p-singular element in the group $Sym(Y)$ of bijections of Y onto itself, and X be the set of fixed points of t. Then p divides $|Y| - |X|$.*

Proof of Proposition 11.18. Let $\varphi_!$ denote the extension of φ to G by 0. Then by the definition of $\mathrm{Ind}_E^G : C(E) \to C(G)$ we have

$$f_a(x) = \sum_{g \in G/E} \varphi_!(g^{-1}xg). \tag{1}$$

Let $x \in G$. If x_r is not conjugate to a then all the terms in the sum are 0 by definition of φ. Assume now that x_r is conjugate to a. Conjugating x we can assume $x_r = a$. It is clear that in the sum (1) above, non-zero contribution is given only by terms g with $(g^{-1}xg)_r = a$. Since $(g^{-1}xg)_r = g^{-1}x_r g = g^{-1}ag$, this implies $g \in Z(a)$. Thus

$$f_a(x) = \sum_{g \in Z(a)/E} \varphi_!(g^{-1}xg). \tag{2}$$

Denote $Y := Z(a)/E$, $X := \{g \in Y \mid g^{-1}x_s g \in S\}$, where x_s is the singular part of x. From (2) we have $f_a(x) = m|X|$. It is left to show that $|X|$ is prime to p. Note that an element $g \in Y$ belongs to X if and only if $x_s g \in gE$. In other words, X is the fixed point set of the left action of x_s on Y. Since $|Y|$ is prime to p, we get that so is $|X|$, by Exercise 11.19. □

12. Representations of topological groups - basic notions

Definition 12.1. *A topological group is a topological space which is also a group such that the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are continuous.*

We will consider only locally compact Hausdorff topological groups. Mostly just compact groups.

Examples of compact groups:

(1) A finite group with discrete topology.
(2) A circle. More generally: $SO(n, \mathbb{R})$ or $O(n, \mathbb{R})$.

Examples of non-compact locally compact groups: $\mathbb{R}$, $\mathbb{C}$, $\mathrm{SL}(2,\mathbb{R}), \mathrm{GL}(n,\mathbb{R}), O(n,\mathbb{C})$.

From now on we fix the ground field F to be $\mathbb{C}$.

Definition 12.2. *A continuous representation of G is a linear representation of G in a Banach space B over $\mathbb{C}$ such that the natural map $G \times V \to V$ is continuous. A morphism of continuous representation is a bounded operator between the corresponding Banach spaces that commutes with the group action.*

Example 12.3. *The regular representation of any compact group K in the Banach space $C(K)$ of continuous functions on K with the maximum norm.*

We can also consider a representation in square-integrable functions, but for that we need a measure.

Theorem 12.4 (Haar)**.** *There exists a unique, up to multiplicative constant, measure on G which is invariant under left shifts.*

This measure is called the Haar measure and denoted by dg.

Corollary 12.5. *(1) There exists a character Δ_G of G, called the modular character, such that $R_g dg = \Delta_G(g)dg$, where R_g denotes the right shift.*
(2) If G is compact, Δ_G is trivial.

Now we can define another regular representation: $L^2(G)$.

Definition 12.6. *A representation is called irreducible if it has no continuous subrepresentations. In other words, every non-zero G-invariant subspace is dense.*

For finite-dimensional continuous representations Schur's lemmas still hold, with the same proofs.

Definition 12.7. *A unitary representation is a representation of G in a Hilbert space H with G-invariant scalar product. A representation is called unitarizable if it is isomorphic to a unitary representation.*

Example 12.8. *$L^2(G)$ is a unitary representation, with the invariant scalar product*

$$\langle f, h\rangle = \int_{g\in G} f(g)\overline{h(g)}dg$$

As in the finite group case, we have:

Lemma 12.9. *Unitary representations are completely reducible.*

From now on, let K be a compact group.

Lemma 12.10. *For any representation (π, B) of K in a Banach space B we have a natural projection $B \to B^K$ - by averaging.*

Lemma 12.11. *Any representation of K in a Hilbert space is unitarizable. In particular, every finite-dimensional representation is unitarizable.*

However, $C(K)$ (continuous functions with maximal norm) is not unitarizable. $L^2(K)$ is unitarizable.

All the statements about finite-dimensional representations of finite groups that we had carry over to the compact case, except, of course, those involving the order of the group. Note that every finite-dimensional vector space has a unique structure of a Hilbert space.

13. The Peter-Weyl theorem and its corollaries

Let $\mathrm{Irr}_f(K)$ denote the set of finite-dimensional irreducible representations. We will later show that these are all the irreducible representations.

The analog of the statement about the decomposition of the regular representation is the Peter-Weyl theorem.

Theorem 13.1 (Peter-Weyl)**.**

$$L^2(K) \simeq \widehat{\bigoplus}_{\rho\in \mathrm{Irr}_f(K)} \mathrm{End}_{\mathbb{C}}(\rho)$$

The map in one direction is defined by matrix coefficients: $M_{\rho,A}(g) = \mathrm{Tr}(A\rho(g^{-1}))$. The action map, in the other direction, is defined only on $C(K)$:

$$\rho(f)v := \int_G f(g)\rho(g)vdg$$

To define the action map, we do not need ρ to be finite-dimensional. I am not sure we will have time to prove this theorem.

In particular, characters of non-isomorphic irreducible representations are orthogonal.

Corollary 13.2. $\bigoplus_{\rho\in \mathrm{Irr}_f(K)} \mathrm{End}(\rho)$ *is dense in* $C(K)$*.*

This follows from the Stone-Weierstrass theorem:

Theorem 13.3 (Stone-Weierstrass)**.** *Let K be a compact (Hausdorff) topological space and $A < C(K)$ be a subalgebra with 1 that separates points and is closed under complex conjugation. Then A is dense in $C(K)$.*

This implies the previous corollary since matrix coefficients form an algebra: sum is given by direct sum, and product by tensor product.

Definition 13.4. *Let (π, B) be a continuous representation of K and ρ be an irreducible finite-dimensional representation. Define a Banach space $M_\rho(\pi) := \mathrm{Hom}_G(\rho, \pi)$ and a continuous representation $\pi_\rho := \rho \otimes \mathrm{Hom}_G(\rho, \pi)$. Note that π_ρ has a natural embedding to π.*

The π_ρ could be zero.

Exercise 13.5. *Let $\rho \in \mathrm{Irr}_f(K)$ and embed $\mathrm{End}_{\mathbb{C}}(\rho)$ into $C(K)$ using the matrix coefficient map. Let π be any continuous representation of K. Then π_ρ is the image of $\mathrm{End}_{\mathbb{C}}(\rho) \otimes \pi$ under the action map $C(K) \otimes \pi \to \pi$ given by $f \otimes v \to \pi(f)v$.*

From the last Corollary we obtain

Corollary 13.6. $\bigoplus_{\rho\in \mathrm{Irr}_f(K)} \pi_\rho$ *is dense in* π*.*

Proof. We can assume that π is generated by one vector. Now, approximate the delta-function by continuous functions, and act on them on this vector. More precisely, we first choose a δ-sequence , *i.e.* a sequence $f_n \in C(K)$ with $\int f_n = 1$ for any n and $\mathrm{Supp} f_n$ shrinking to $\{1\} \subset K$. Next, one can show that for any vector v, $\pi(f_n)v \to v$. This shows that the map $C(K) \otimes \pi \to \pi$ given by $f \otimes v \mapsto \pi(f)v$ has dense image. By the previous corollary, its restriction to $(\bigoplus_{\rho\in \mathrm{Irr}_f(K)} \mathrm{End}_{\mathbb{C}}(\rho)) \otimes \pi$ still has a dense image. By the previous exercise, this image is $\bigoplus_{\rho\in \mathrm{Irr}_f(K)} \pi_\rho$. ∎

Corollary 13.7. *All irreducible representations of K are finite-dimensional.*

Proof. Let π be an irrep of K. By the previous corollary, $\pi_\rho \neq 0$ for some $\rho \in \mathrm{Irr}_f(K)$. Thus the multiplicity space $M_\rho(\pi) := \mathrm{Hom}_G(\rho, \pi)$ does not vanish. Thus ρ embeds into π. Since π is irreducible, this embedding has a dense image. Since finite-dimensional subspaces of Banach spaces are closed, this implies that $\pi \cong \rho$, and thus π is finite-dimensional. $\square$

Corollary 13.8. *For any π and irreducible ρ, we have a natural projection $\pi \twoheadrightarrow \pi_\rho$, given by $\pi(\chi_\rho)$.*

Thus, we have $\bigoplus_{\rho \in \mathrm{Irr}(K)} \pi_\rho \subset \pi \subset \prod_{\rho \in \mathrm{Irr}(K)} \pi_\rho$. This implies

Corollary 13.9. *If (π, H) is a unitary representation then $\pi = \widehat{\bigoplus}_{\rho \in \mathrm{Irr}(K)} \pi_\rho$.*

However, for Banach space representations we do not have such a decomposition, even for $C(S^1)$.

One can define induction $\mathrm{Ind}_H^G(\pi)$ in a similar to the finite group case: consider H-equivariant continuous functions from G to π. If π is unitary, one can also consider a "unitary induction": square-integrable functions from G to π. This will be a unitary representation. The proper notion of equivariant sheaf is missing in general, but Mackey theory holds for unitary inductions of unitary representations.

If G/H is not compact, one can also consider a "small induction": continuous functions from G to π with compact support modulo H. This case is quite difficult to study, so people prefer to consider co-compact subgroups, for example the subgroup of upper-triangular matrices in $\mathrm{GL}(n, \mathbb{R})$.

If H is compact, one has a nice theory of Gelfand pairs. If not, one can also say something, but it becomes very delicate. I have several results in this case.

14. Harmonic analysis on the sphere and an application to integral geometry

Let $S^{n-1} \subset \mathbb{R}^n$ denote the sphere of radius 1 and center at the origin. The group $SO(n)$ of rotations in $\mathbb{R}^n$ acts transitively on S^{n-1}, and the stabilizer of a point is $SO(n-1)$. Harmonic analysis on S^{n-1} means the study of $L^2(S^{n-1})$ as a representation of $SO(n)$. The case $n = 2$ is the Fourier analysis, and in this section we generalize it. We will find the decomposition of $L^2(S^{n-1})$ to irreducible representations and use it to prove the following theorem.

Theorem 14.1. *Every closed convex central-symmetric body in $\mathbb{R}^n$ is uniquely determined by the areas of its projections on all hyperplanes.*

This theorem is equivalent to the following one:

Theorem 14.2. *Every closed convex central-symmetric body in $\mathbb{R}^n$ is uniquely determined by the areas of its intersections with all hyperplanes (passing through the origin).*

Let us show their equivalence.

Definition 14.3. *Call two closed convex central-symmetric bodies $K, K' \subset \mathbb{R}^n$ dual if*
$$\sup_{y\in K'} \langle x, y\rangle \leq 1 \Leftrightarrow x \in K,$$
where $\langle x, y\rangle$ denotes the standard scalar product in $\mathbb{R}^n$. Note this condition is equivalent to the condition
$$\sup_{x\in K}\langle x, y\rangle \leq 1 \Leftrightarrow y \in K'.$$

The equivalence now follows from the following exercise.

Exercise 14.4. *Let $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ be a hyperplane and p denote the projection to $\mathbb{R}^{n-1}$. Show that if K is dual to K' in $\mathbb{R}^n$ then $\mathbb{R}^{n-1} \cap K$ is dual to $p(K')$ in $\mathbb{R}^{n-1}$.*

Let us now prove Theorem 14.2. For simplicity, take $n = 3$ and denote $S := S^2 \subset \mathbb{R}^3$. For any convex central-symmetric body K, define a function f_K on S by $f_K(x) = 1/2r_x^2$, where r is the length of the segment which is the intersection of K with the line passing through the origin and x. Note that f_K is an even function which completely determines K.

Exercise 14.5. *Let $P \subset \mathbb{R}^3$ be a plane. Then*
$$Area(K \cap P) = \int_{S\cap P} f_K(x)dx$$

Thus, Theorem 14.2 follows from the statement that an even function on the sphere is uniquely determined by its integrals on all the big circles. Denote by $L_+(S)$ the subrepresentation consisting of even functions, and by J the morphism $L^2(S) \to L^2_+(S)$ given by
$$Jf(x) := \int_{C_x} f(y)dy,$$
where C_x denotes the big circle with epicenter in x. By the Peter-Weyl theorem and Schur's lemmas, we know that $L_+(S)$ is a completed direct sum or irreducible representations and J is scalar on each summand. Let us find this decomposition.

Denote by P_n the space of all functions on S that are restrictions of polynomials of degree n in $\mathbb{R}^3$.

Exercise 14.6. $P_n \subset P_{n+2}$ *and* $\dim P_n = (n + 1)(n + 2)/2$.

Let H_n denote the orthogonal complement to P_{n-2} in P_n (under the natural scalar product in $L^2(S)$).

Remark 14.7. *One can identify H_n with the space of homogeneous harmonic polynomials of degree n in $\mathbb{R}^3$. Harmonic means that the vanish under the Laplace operator*
$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$
Thus, the functions in H_n are called 'spherical harmonics'.

However, we will not use this identification.

Lemma 14.8.
$$L^2(S) = \widehat{\bigoplus}_{n=0}^{\infty} H_n, \quad L^2_+(S) = \widehat{\bigoplus}_{n=0}^{\infty} H_{2n}$$

Proof. Clearly, H_n are invariant, orthogonal and their sum is the union of all P_n. This union separates points of S, thus, by the Stone-Weierstrass theorem, is dense in $C(S)$ and thus in $L^2(S)$. Clearly, $H_n \subset L_+(S)$ if and only if n is even. □

Let us now show that H_n is irreducible. Let $SO(2) \subset SO(3)$ denote the subgroup of rotations with respect to the z axis and identify $S = SO(3)/SO(2)$.

Exercise 14.9. *Show that* $\dim(P_n)^{SO(2)} = [n/2] + 1$

Hint. Show that $(P_n)^{SO(2)}$ is spanned by z^n, $z^{n-2}(x^2+y^2)$, ..., $z^{n-2[n/2]}(x^2+y^2)^{[n/2]}$. □

Now, note that by Frobenius reciprocity every irreducible subrepresentation of $L^2(S)$ has an $SO(2)$-invariant vector. This proves

Lemma 14.10. *H_n are irreducible.*

This finishes the harmonic analysis problem. To prove the integral geometry theorem, it is left to compute the eigenvalues of J. For this we can pick any function in each H_n that is convenient to us. We choose the $SO(2)$-invariant function, which is also called the n-th Legandre polynomial:

$$L_n(z) = \frac{d^n}{dz^n}((z^2-1)^n).$$

Exercise 14.11. $L_n \in H_n$.

Hint. Show that for any $SO(2)$-invariant function f on S we have $\int_S f(x)dx = \int_{-1}^1 2\pi z f(z)dz$, deduce that $\langle f_1, f_2\rangle = \int_{-1}^1 2\pi z f_1(z)\overline{f_2(z)}dz$ and use integration by parts to show that $L_n(z)$ is orthogonal to all polynomials in z of degree smaller than n. □

Now, let λ_n be the eigenvalue of J on H_n. Then $JL_n = \lambda_n L_n$ and in particular $JL_n(1) = \lambda_n L_n(1)$. In addition, from the definitions of J and of L_n we see that $JL_n(1) = 2\pi L_n(0)$. Altogether, we get

$$\lambda_n L_n(1) = 2\pi L_n(0).$$

The values $L_n(1)$ and $L_n(0)$ are easy to compute:

$$L_n(1) = -\frac{d^n}{dz^n}((z-1)^n(z+1)^n)|_{z=1} = n!2^n,$$

$$L_{2k+1}(0) = 0, \quad L_{2k}(0) = (2k)!\binom{2k}{k}.$$

Thus,

$$\lambda_{2k+1} = 0 \text{ and } \lambda_{2k} = 2\pi\frac{(2k-1)!!}{(2k+1)!!}.$$

This gives an explicit formula for the inverse of J on $L_+(S)$ and proves Theorem 14.2.

Remark 14.12. *Theorem 14.2 and the proof we discussed generalizes to higher dimensions. However, for $S^2 \subset \mathbb{R}^3$ there is one special property: every irreducible representation of $SO(3)$ is isomorphic to one of the H_n. Thus we have a classification of all irreducible representations of $SO(3)$. Also, we get that $L^2(S^2)$ includes each irreducible representation exactly one time (unlike $L^2(SO(3))$ which includes each π $\dim\pi$ times). Such representations are called "models".*

15. Proof of the Peter-Weyl theorem

Recall that the theorem states

$$L^2(K) \simeq \widehat{\bigoplus}_{\rho \in Irr_f(K)} \mathrm{End}_{\mathbb{C}}(\rho)$$

The map in one direction is defined by matrix coefficients: $M_{\rho,A}(g) = \mathrm{Tr}(A\rho(g^{-1}))$. The action map, in the other direction, is defined only on $C(K)$:

$$\rho(f)v := \int_K f(g)\rho(g)vdg$$

Let (ρ, V) be a finite-dimensional continuous irreducible representation of K and let $C_\rho \subset L^2(K)$ denote the image of the matrix coefficients map $M_\rho : \mathrm{End}_{\mathbb{C}}(V) \to C(K)$. Note that $\mathrm{End}_{\mathbb{C}}(V)$ is an irreducible representation of $K \times K$, thus M_ρ has no kernel and thus defines an isomorphism $\mathrm{End}_{\mathbb{C}}(V) \simeq C_\rho$. Note also that as a representation of K, C_ρ is isotypic of type ρ and thus C_ρ and C_σ are orthogonal for $\rho \not\cong \sigma$. It is left to show that $\bigoplus_{\rho \in \mathrm{Irr}_f(K)} C_\rho$ is dense in $L^2(K)$. We do that in several steps.

Lemma 15.1. *Every subrepresentation $W \subset L^2(K)$ isomorphic to ρ lies inside C_ρ.*

Proof. We can suppose that all the functions in W have value at $1 \in G$. Indeed, every L^2 function has a value almost everywhere. Choose a basis for W. It will be finite, and thus there will be a point $k \in K$ at which all the basis elements have a value. Acting by k^{-1} we obtain another basis, consisting of functions that have values at 1. Thus, so do all their linear combinations, *i.e.* all the vectors in W.

Now, evaluation at 1 defines a linear functional $\delta_1 \in W^*$. Consider $f \otimes \delta_1 \in \mathrm{End}_{\mathbb{C}}(W) \simeq \mathrm{End}_{\mathbb{C}}(\rho)$ and note that $f = M_{f\otimes\delta_1} \in C_\rho$. □

Lemma 15.2. *Every non-zero subspace of $L \subset L^2(K)$ which is invariant under $K \times K$ has a non-zero finite-dimensional subspace which is invariant under the left action of K.*

This is the hardest lemma in the proof, and it uses spectral theory for compact self-adjoint operators on Hilbert spaces. That it, we will use the following definition and theorems.

Definition 15.3. *Let B be a Banach space, and $A : B \to B$ be a continuous linear operator. A is called compact if the image of the unit ball under A is compact.*

Definition 15.4. *Let K be a compact topological space, and $F \in C(K \times K)$. Then F defines an continuous operator $A_F : L^2(K) \to L^2(K)$ by $A_F(h)(k) := \int_{x\in K} F(k,x)h(x)dx$. We say that the operator A_F is given by the kernel F.*

Theorem 15.5. *For any kernel F, the operator A_F is compact.*

Theorem 15.6 (Fredholm)**.** *(i) For any compact operator on a Banach space, all eigenspaces corresponding to non-zero eigenvalues are finite-dimensional.*

(ii) Any compact self-adjoint operator on a Hilbert space H is diagonalizable. That is, the space decomposes to a completed direct sum of eigenspaces.

The reason for (i) is that on infinite-dimensional spaces, unit balls are not compact, and thus non-zero scalar operators are not compact operators.

Proof of Lemma 15.2. Consider the right action of K on $L^2(K)$. Since the left and the right actions commute, this defines an intertwining operator $R(f) : L^2(K) \to L^2(K)$ for any $f \in L^2(K)$. This operator is compact (since it is given by a compact kernel). The adjoint operator is $R(f^*)$ where $f^*(g) = \overline{f(g^{-1})}$. Now, for any $v \in L$ we can find $f \in L^2(K)$ such that $f = f^*$ and $R(f)v \neq 0$. Then $R(f)$ is compact and self-adjoint and thus $\overline{L}$ can be decomposed to a completed direct sum of eigenspaces of $R(f)$, and all eigenspaces except the kernel are finite-dimensional. Since $R(f)$ is non-zero, there exists a non-zero finite-dimensional eigenspace. It is invariant under the left action of K, since this action commutes with $R(f)$. $\square$

Those two lemmas imply that $\bigoplus_{\rho \in \mathrm{Irr}_f(K)} C_\rho$ is dense in $L^2(K)$. Indeed, suppose it is not dense. Then it has a non-zero orthogonal complement L'. By Lemma 15.2, L' has a finite-dimensional subrepresentation W. Then W has a subrepresentation (ρ, V) for some $\rho \in \mathrm{Irr}_f(K)$. By Lemma 15.1, $V \subset C_\rho$, but by definition V is orthogonal to C_ρ - contradiction. $\square$

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URL: http://www.wisdom.weizmann.ac.il/~dimagur/IntRepTheo4.html