INTRODUCTION TO COMPUTATION OF DERIVED FUNCTORS USING RESOLUTIONS, AND SHEAF COHOMOLOGY

Derived functors allow to compute to which extent functors are not exact, and allow to compensate for this non-exactness. We use resolutions in order to compute derived functors. For this we need a subclass of objects, on which the derived functor is concentrated at the zero cohomology. Such objects are called *acyclic*
The original functor is exact on sequences of acyclic objects. Projective objects are always acyclic for right-exact functors, and injective objects are always acyclic for left-exact functors. But many times there are more acyclic objects. For example, modules acyclic w.r.to tensor product are called *flat*, and there exist non-projective flat modules, for example $K[t, t^{-1}]$ over $K[t]$.
A very important left-exact functor is the functor Γ of global sections on sheaves. The cohomologies of its derived functor are called *sheaf cohomologies*: $H^i(X, \mathcal{F}) := H^i(R\Gamma(\mathcal{F}))$. In fact, sheaves were invented to compute cohomologies of topological spaces. These cohomologies can be computes using constant sheaves:

$$H^i(X, \mathbb{C}) \cong H^i(X, \mathbb{C}_X)$$

Here, $\mathbb{C}_X(U) = \{\text{locally-constant functions } U \to \mathbb{C}\}$.
There are several kinds of acyclic sheaves. They include *flabby* sheaves, i.e. sheaves for which the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is onto for any $U \subset V$.

Exercise 1. *Every injective sheaf is flabby. Hint: use* $(j_V)_\bullet \mathcal{F}|_V$.

One can always construct a resolution with flabby sheaves. It is called the Godement resolution. For any sheaf $\mathcal{F}$, define $G_\mathcal{F}$ by $G_\mathcal{F}(U) := \prod_{u \in U} \mathcal{F}_u$. This sheaf is clearly flabby, and we have a canonical embedding $\mathcal{F} \hookrightarrow G_\mathcal{F}$. Here, $\mathcal{F}_u$ denotes the fiber. It consists of equivalence

Date: January 12, 2021.

classes of pairs $\{(V,\xi)\,|\,u\in U, \xi\in\mathcal{F}(V)\}/\sim$ where $(V,\xi)\sim(V',\xi')$ iff $\xi|_{V\cap V'}=\xi'|_{V\cap V'}$. A sequence of sheaves is exact if and only if it is exact in every stalk.

Another type of acyclic sheaves are sheaves with partition of unity. This means that for every open $U\subset X$ and every open cover $\bigcup_i U_i=U$, and every section $\xi\in\mathcal{F}(U)$, there exist sections $\xi_i\in\mathcal{F}(U)$ such that $\xi=\sum\xi_i$ and $\mathrm{supp}\xi_i\subset U_i$. Here, $\mathrm{supp}(\xi)$ is the complement in U to the union of all open $V\subset U$ with $\xi|_U=0$.

Thus, the de-Rham complex is the resolution of the constant sheaf. That is, for every smooth manifold X, the following sequence is exact:

$$0\to\mathbb{C}_X\to C^\infty_X\to\Omega^1_X\to\cdots\to\Omega^n_X\to 0$$

This can be shown locally, and for $\mathbb{R}^n$ it is Poincare's lemma. The proof of Poincare's lemma is using the Newton-Leibnitz formula, and the fact that every cycle in $\mathbb{R}^n$ is a boundary (i.e. the fact that all homologies of $\mathbb{R}^n$ vanish). The latter holds because $\mathbb{R}^n$ is contractible. Thus, the global sections of this complex compute the cohomologies of X.

For an algebraic variety X, one computes cohomologies of sheaves using the Chech complex. The Chech complex uses the fact that Γ is exact on affine varieties. Thus, the sheaves $(j_U)_\bullet\mathcal{O}_U$ are acyclic for affine open subsets $U\subset X$. The sheaf axioms give us an embedding $\mathcal{O}_X\hookrightarrow\bigoplus_U(j_U)_\bullet\mathcal{O}_U$. If X is separated, then we have the Chech resolution

$$0\to\mathcal{O}_X\to\bigoplus_U(j_U)_\bullet\mathcal{O}_U\to\bigoplus_{U,V}(j_{U\cap V})_\bullet\mathcal{O}_{U\cap V}\to\cdots\to(j_{\cap U})_\bullet\mathcal{O}_{\cap U}\to 0$$

Finally, let us give an example in which Γ is not right-exact (and thus one of the sheaves is not acyclic):

Let $X:=\mathbb{A}^2\setminus\{0,0\}$ and $Z:=\mathbb{A}^1\setminus\{0,0\}$. Then Z is affine while X is not, and $\mathcal{O}_X(X)=\mathbb{K}[x,y]$, $\mathcal{O}_Z(Z)=\mathbb{K}[x,x^{-1}]$. We have an onto map $\mathcal{O}_X\twoheadrightarrow i_\bullet\mathcal{O}_Z$. But on global sections it is not onto! Thus we have $\mathcal{O}_X(X)\to\mathcal{O}_Z(Z)\to H^1(X,K)$, where $K:=Ker(\mathcal{O}_X\twoheadrightarrow i_\bullet\mathcal{O}_Z)$. Thus K is not acyclic.