Stone-von Neumann equivalence for smooth representations of the Heisenberg group

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July 25, 2024

The classical Stone-von Neumann theorem

Let H be a Heisenberg group with center Z, *i.e.*

$$H = W \times Z,$$

where W is a symplectic space. Let $L \subset W$ be Lagrangian subspace, and let $P := L \times Z$. Let $\chi \neq 1 \in \widehat{Z}$, extend χ to L trivially. Let $\Omega_\chi = \mathrm{ind}_P^H \chi$.

Theorem (Stone-von Neumann)

Let τ be an irreducible unitary representation of H such that

$$\tau(z)v = \chi(z)v, \qquad \text{for all } z \in Z.$$

Then $\tau \cong \Omega_\chi$.

Geometric basis for this theorem: denote $\widehat{P}_\chi := \{\psi \in \widehat{P} \mid \psi|_Z = \chi\}$. Then $\widehat{P}_\chi = H/P$ and the theorem follows from Mackey imprimitivity thm.

Our version

$H = W \times Z$, $L \subset W$, $P := L \times Z$. Let $\widehat{Z}^{\times} := \widehat{Z} \setminus \{1\} \simeq \mathbb{R}^{\times}$.
Let $\mathrm{Rep}^{\infty}(H)$ and $\mathrm{Rep}^{\infty}(Z)$ denote the categories of smooth Fréchet representations of moderate growth.
We define subcategory $\mathrm{Rep}^{\infty}(Z)^{\times} \subset \mathrm{Rep}^{\infty}(Z)$ of representations "supported on non-trivial characters". $\mathrm{Rep}^{\infty}(H)^{\times}$:=subcategory of $\mathrm{Rep}^{\infty}(H)$ consisting of representations π *s.t.* $\pi|_Z \in \mathrm{Rep}^{\infty}(Z)^{\times}$. Let

$$C : \mathrm{Rep}^{\infty}(H)^{\times} \leftrightarrows \mathrm{Rep}^{\infty}(Z)^{\times} : I \quad C(\tau) := \tau_L,\ I(\rho) := \mathrm{ind}_P^H \rho$$

Theorem (Gomez-G.-Sahi '24)

The functors I and C are mutually quasi-inverse equivalences of categories.

Formal def. of $\mathrm{Rep}^{\infty}(Z)^{\times}$: Let $\mathfrak{z} := Lie(Z), z \neq 0 \in \mathfrak{z}$.

$$\mathrm{Rep}^{\infty}(Z)^{\times} := \{\rho \in \mathrm{Rep}^{\infty}(Z) \,|\, \exists \sigma \in \mathrm{Rep}^{\infty}(Z) \text{ s.t. } d\rho(z)d\sigma(z) = \mathrm{Id}\}$$

Idea of proof

- Let $G \curvearrowright X$. du-Cloux: a G-imprimitivity system $\mathcal{F}$ on a G-space X is a Fréchet space with compatible structures of a module over the algebras $\mathcal{S}(X)$ (with pointwise multiplication as product) and on $\mathcal{S}(G)$ (with convolution).
- Geometrically: G-imprimitivity system = equivariant sheaf: $\forall x \in X$ have $\mathcal{F}_x$, and $\forall g \in G$ a continuous linear operator $\mathcal{F}_x \to \mathcal{F}_{gx}$. The space $\mathcal{F}$ is the space of global sections of this sheaf.
- Let $\tau \in \mathrm{Rep}^\infty(H)^\times$. Using Fourier transform on $L \times Z$, τ defines an H-imprimitivity system $\mathcal{F}$ on $\widehat{L} \times \widehat{Z}$, with the property that $L \times Z \subset H$ acts on $\mathcal{F}_{(\chi,\psi)}$ by the character (χ, ψ), $\forall (\chi, \psi) \in \widehat{L} \times \widehat{Z}$.
- By defn of $\mathrm{Rep}^\infty(H)^\times$, $\mathcal{F}$ is completely determined by $\mathcal{F}|_{\widehat{L} \times \widehat{Z}^\times}$.
- Since $0 \times \widehat{Z}^\times \subset \widehat{L} \times \widehat{Z}^\times$ is a section transversal to the action of H, the stabilizer of any point $(\chi, \psi) \in \widehat{L} \times \widehat{Z}$ is $L \times Z$, and the action on $\mathcal{F}_{(\chi,\psi)}$ by (χ, ψ), the system really contains the same information as a sheaf on $\widehat{Z}^\times$, or equivalently a representation $\rho \in \mathrm{Rep}^\infty(Z)^\times$.

Towards a generalized Segal-Shale-Weil

Define $\varpi \in \mathrm{Rep}^{\infty}(Z)^{\times}$ by $\varpi := \mathcal{S}(\widehat{Z}^{\times})$ with $f^{z}(\chi) := \chi(z)f(\chi)$.

Lemma

$\forall V \in \mathrm{Rep}^{\infty}(Z)^{\times}$ we have $\varpi \widehat{\otimes} V \twoheadrightarrow V$.

Define $\Omega := \mathrm{ind}_{P}^{H} \omega \in \mathrm{Rep}^{\infty}(H)^{\times}$. Then $\Omega_{\chi}^{\infty} = \Omega_{Z,\chi}$.

Lemma

$\forall \tau \in \mathrm{Rep}^{\infty}(H)^{\times}, \rho \in \mathrm{Rep}^{\infty}(Z)^{\times}$ have

$$C(\tau) := \tau_{L} \cong \tau \widehat{\otimes}_{H} \overline{\Omega} := (\tau \widehat{\otimes} \overline{\Omega})_{H}, \; I(\rho) := \mathrm{ind}_{P}^{H} \rho \cong \rho \widehat{\otimes}_{Z} \Omega := (\rho \widehat{\otimes} \Omega)_{Z}$$

Proposition (Gomez-G.-Sahi '24)

There is a unique extension of Ω to a representation of $\widetilde{\mathrm{Sp}}(W) \ltimes H$.

Generalized Segal-Shale-Weil representation

- $\varpi \in \mathrm{Rep}^\infty(Z)^\times$ by $\varpi := \mathcal{S}(\widehat{Z}^\times)$ with $f^z(\chi) := \chi(z)f(\chi)$.
- $\Omega := \mathrm{ind}_P^H \omega \in \mathrm{Rep}^\infty(H)^\times$ realized in functions $W \to \varpi$.
- $n := \dim W/2$. Introduce a basis on W s.t. L is spanned by the last n basis vectors, and the symplectic form is given in this basis by

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

- Identify the Lagrangian spanned by the first n coordinates with $R \cong W/L$.
- Define an operator S on the space of ϖ by

$$(S\phi)(\chi_s) := \sqrt{|s|}\phi(\chi_s), \text{ where } \chi_s(t) := \exp(2\pi ist)$$

- Define analogues of Fourier transform and its inverse on Ω by

$$\mathcal{F}_\varpi(f)(x) := S^{-1}\int_L \varpi(-x^t y)f(y)dy \quad \text{and}$$
$$\mathcal{F}_\varpi^{-1}(f)(x) := S\int_L \varpi(x^t y)f(y)dy.$$

Generalized Segal-Shale-Weil representation

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- $(S\phi)(\chi_s) := \sqrt{|s|}\phi(\chi_s)$, where $\chi_s(t) := \exp(2\pi ist)$
- $\mathcal{F}_\varpi^{-1}(f)(x) := S \int_L \varpi(x^t y) f(y) dy$.

Define $\widetilde{\mathrm{Sp}} \curvearrowright \Omega$ by:

- $\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix} f(x) := \pm(\det A)^{-1/2} f(A^{-1}x)$
- $\begin{pmatrix} \mathrm{Id} & 0 \\ C & \mathrm{Id} \end{pmatrix} f(x) := \pm\varpi(-x^t Cx/2) f(x)$
- $\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} f(x) :=\mapsto \pm i^{n/2} \mathcal{F}_\varpi^{-1}(f)(x)$

Proposition (Gomez-G.-Sahi '24)

This defines the unique extension of Ω to a representation of $\widetilde{\mathrm{Sp}}(W) \ltimes H$.

Fourier-Jacobi equivalence

$$\mathrm{Rep}^\infty(\widetilde{\mathrm{Sp}}(W) \ltimes H)^\times_- :=$$
$$\{\text{ genuine } \tau \in \mathrm{Rep}^\infty(\widetilde{\mathrm{Sp}}(W) \ltimes H) \text{ s.t. } \tau|_H \in \mathrm{Rep}^\infty(H)^\times\}.$$

$$C^+ : \mathrm{Rep}^\infty(\widetilde{\mathrm{Sp}}(W) \ltimes H)^\times_- \leftrightarrows \mathrm{Rep}^\infty(\mathrm{Sp}(W) \times Z)^\times : I^+$$

- $C^+(\tau) := \tau \widehat{\otimes}_H \overline{\Omega}$ with $\widetilde{\mathrm{Sp}}(W)$ acting diagonally, and Z acting on τ.
- $I^+(\rho) := \rho \widehat{\otimes}_Z \Omega$, with H acting on Ω, and $\widetilde{\mathrm{Sp}}$ acting diagonally.

Theorem (Gomez-G.-Sahi '24)

The functors C^+, I^+ are quasi-inverses.

$$T : \mathrm{Rep}^\infty(H)^\times \xrightarrow{C} \mathrm{Rep}^\infty(Z)^\times \xrightarrow{\text{triv}} \mathrm{Rep}^\infty(\mathrm{Sp}(W) \times Z)^\times \xrightarrow{I^+} \mathrm{Rep}^\infty(\widetilde{\mathrm{Sp}}(W) \ltimes H)^\times_-$$

Theorem (Gomez-G.-Sahi '24)

The functor T is the unique right quasi-inverse of the restriction functor

$$\mathrm{Rep}^\infty(\widetilde{\mathrm{Sp}}(W) \ltimes H)^\times_- \to \mathrm{Rep}^\infty(H)^\times.$$

Examples

Let $\dim W = 2$. Let $\rho_i :=$ Taylor expansion by s of $\exp(2\pi ist)$ at $s = 1$, of order i.

Example ($i = 2$)

ρ is two-dimensional, with $\rho(t) = \begin{pmatrix} \exp(2\pi it) & 2\pi it\exp(2\pi it) \\ 0 & \exp(2\pi it) \end{pmatrix}$

The action of S in this case is $\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$ and

$\sigma \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} = \pm \begin{pmatrix} 1 & E \\ 0 & 1 \end{pmatrix} \mathcal{F}^{-1}$, where $E = (x\partial + \partial x)/2 = x\partial + 1/2$ is the symmetrized Euler operator, and $\mathcal{F}^{-1} = \mathcal{F}_1^{-1}$ is the classical inverse Fourier transform.

Examples

Example ($i = 3$)

$$\rho(t) = \begin{pmatrix} \exp(2\pi it) & 2\pi it\exp(2\pi it) & -2\pi^2t^2\exp(2\pi it) \\ 0 & \exp(2\pi it) & 2\pi it\exp(2\pi it) \\ 0 & 0 & \exp(2\pi it) \end{pmatrix}$$

The action of S in this case is $\begin{pmatrix} 1 & 1/2 & -1/4 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$. Also, in this case

$$\sigma(w = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}) = \pm i^{1/2} \begin{pmatrix} 1 & E & E^2/2 - E/2 \\ 0 & 1 & E \\ 0 & 0 & 1 \end{pmatrix} \mathcal{F}^{-1}.$$

Since $w^2 = -\mathrm{Id}$, this operator must square to $f(x) \mapsto if(-x)$.
If we do not do Fourier expansion then, squaring the operator for "Fourier transform", we get $F(x,s) \mapsto iF(-x,s)$. Deriving this identity in the variable s, we get generalizations of the previous examples to any order.

Infinitesimal Weil representation

Action of the Lie algebra $\mathfrak{sl}_2 = \mathrm{Span}(X, H, Y)$: We have

$$\sigma(Y)F(x,s) = \frac{\partial}{\partial c}\exp(-2\pi i s c x^2/2)F(x,s)|_{c=0} = -\pi i s x^2 F(x,s).$$

$$\sigma(H)F(x,s) = \frac{\partial}{\partial c}\exp(-c/2)F(\exp(-c)x,s)|_{c=0} =$$
$$-F(x,s)/2 - x\partial_x F(x,s) = -EF(x,s).$$

Conjugating the action of Y by $\sigma(\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix})$ we get

$$\sigma(X)(F)(x,s) = -\pi i s\mathcal{F}(x^2\mathcal{F}^{-1}(F(x,s)))(-xs),s) = (4\pi i s)^{-1}\partial_x^2 F(x,s).$$

Since $[E, x^2] = 2x^2, [E, \partial_x^2] = -2\partial_x^2$, and $[\partial_x^2, x^2] = 4E$, we get that $\sigma(X), \sigma(H), \sigma(Y)$ form an $\mathfrak{sl}_2$-triple. If we substitute $s := 1$ we obtain the formulas for the classical infinitesimal Weil representation.

Action of the Lie algebra $\mathfrak{sl}_2 = \mathrm{Span}(X, H, Y)$: We have

$$\sigma(Y)F(x,s) = \frac{\partial}{\partial c}\exp(-2\pi iscx^2/2)F(x,s)|_{c=0} = -\pi isx^2F(x,s).$$

$$\sigma(H)F(x,s) = \frac{\partial}{\partial c}\exp(-c/2)F(\exp(-c)x,s)|_{c=0} =$$
$$-F(x,s)/2 - x\partial_x F(x,s) = -EF(x,s).$$

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Happy birthday, Chenbo!