THE JET SCHEMES OF THE NILPOTENT CONE OF $\mathfrak{gl}_n$ OVER $\mathbb{F}_\ell$ AND ANALYTIC PROPERTIES OF THE CHEVALLEY MAP

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Abstract. We prove dimension bounds on the jet schemes of the variety of nilpotent matrices (and of related varieties) in positive characteristic.

This result has applications to the analytic properties of the Chevalley map $p : \mathfrak{gl}_n \to \mathfrak{c}$ that sends a matrix to its characteristic polynomial. We show that our dimension bound implies, under the assumption of existence of resolution of singularities in positive characteristic, that the Chevalley map pushes a smooth compactly supported measure to a measure whose density function is L^t for any $t < \infty$.

We also prove this analytic property of the Chevalley map, unconditionally, when the characteristic of the field exceeds $\frac{n}{2}$.

The zero characteristic counterpart of this result is an important step in the proof of the celebrated Harish-Chandra's integrability theorem. In a sequel work [AGKSb] we show that also in positive characteristic, this analytic statement implies Harish-Chandra's integrability theorem for cuspidal representations of the general linear group.

Contents

Date: February 18, 2026.

2020 *Mathematics Subject Classification.* 14L30,20G25,28C15,14B05, 14B10,14E15.

Key words and phrases. rational singularities, smooth measures, resolution of singularities, Chevalley map, jet schemes, reductive group, positive characteristic.

1. Introduction

1.1. **Results on dimensions of jet schemes.** Fix a finite field $\mathbb{F}_\ell$. Unless explicitly stated otherwise, all the algebraic varieties that we consider will be defined over $\mathbb{F}_\ell$. For a variety $\mathbf{X}$ we denote by $\mathcal{J}_m(\mathbf{X})$ its m-th jet scheme. We consider $\mathcal{J}_m$ as a functor from the category of varieties to the category of schemes. We fix an integer n and set $\underline{\mathfrak{g}} := \mathfrak{gl}_n$ considered as an algebraic variety.

In this paper we prove:

Theorem A (§8)**.** *Let $\mathbf{N} \subset \underline{\mathfrak{g}}$ be the nilpotent cone. There is a constant C_0 such that for any $m \in \mathbb{N}$ we have*

$$\dim \mathcal{J}_m(\mathbf{N}) < m \dim(\mathbf{N}) + C_0.$$

We deduce from this result bounds on jet schemes of more varieties. To formulate these bounds we make:

Notation 1.1.1. *Denote by*

- $\underline{\mathfrak{c}}$ *- the affine space of monic polynomials of degree n. We will identify it with $\mathbb{A}^n$.*
- $p : \underline{\mathfrak{g}} \to \underline{\mathfrak{c}}$ *- the Chevalley map (essentially sending an element to its characteristic polynomial).*

- *For an integer $i \in \mathbb{N}$ we denote by $\underline{\mathfrak{g}}_i := \underbrace{\underline{\mathfrak{g}} \times_{\underline{\mathfrak{c}}} \dots \times_{\underline{\mathfrak{c}}} \underline{\mathfrak{g}}}_{i \text{ times}}$ the i-folded fiber product of $\underline{\mathfrak{g}}$ with itself over $\underline{\mathfrak{c}}$ with respect to the map p.*[1]

We deduce the following:

Theorem B (§8). *There is a constant C such that for any $x \in \underline{\mathfrak{c}}$ and any $m \in \mathbb{N}$ we have*

$$\dim \mathcal{J}_m(p^{-1}(x)) < m \dim(p^{-1}(x)) + C.$$

From this we deduce the following:

Theorem C (§8). *For any i there is a constant C_i such that for any $m \in \mathbb{N}$ we have*

$$\dim \mathcal{J}_m(\underline{\mathfrak{g}}_i) < m \dim(\underline{\mathfrak{g}}_i) + C_i.$$

1.2. **Results on pushforward of measures.** We deduce from the results above the following one.

Theorem D (§11). *Let $i \in \mathbb{N}$. Assume that the variety $\underline{\mathfrak{g}}_i$ admits a strong resolution of singularities. Let $\mu^{\mathfrak{c}}$ be a Haar measure on $\mathfrak{c}$.*

Then for any smooth compactly supported measure μ on $\mathfrak{g} := \underline{\mathfrak{g}}(F)$, there exists a function

$$f \in \bigcap_{t \in [1,i)} L^t(\mathfrak{c})$$

such that $p_*(\mu) = f\mu^{\mathfrak{c}}$.

Remark 1.2.1. *In §12 we give several alternative conditions on resolution of singularities under which the result holds.*

Finally we show that one can replace the assumption of Theorem D on the existence of resolution with an assumption on characteristic:

Theorem E (§13). *Suppose* $\operatorname{char}(\mathbb{F}_\ell) > \frac{n}{2}$. *Let $F := \mathbb{F}_\ell((t))$.*

Then for any smooth compactly supported measure μ on $\mathfrak{g} := \underline{\mathfrak{g}}(F)$, the measure $p_*(\mu)$ can be written as a product of a function in $L^\infty(\mathfrak{c})$ and a Haar measure on $\mathfrak{c}$.

1.3. **Background and motivation.**

1.3.1. *FRS maps.* Theorems D and E are related to the notion of FRS maps introduced and studied in [AA16]. Let us recall this notion:

Definition 1.3.1. *A map $\phi : \mathbf{X} \to \mathbf{Y}$ of smooth algebraic varieties over a field of characteristic zero is called FRS if it is flat, its fibers are reduced, and the singularities of its fibers are rational.*

The motivation to this definition is the following:

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[1] A-priory this is a scheme, but we will see in Lemma 5.0.10 below that it is reduced, so it is a variety.

Theorem 1.3.2 ([AA16, Theorem 3.4], [Rei18]). *Let ϕ be a map of smooth algebraic varieties over a local field F of characteristic zero.*

If ϕ is FRS then for any smooth compactly supported measure μ on $\mathbf{X}(F)$, the measure $\phi_*(\mu)$ on $Y := \mathbf{Y}(F)$ can be written as a product of a continuous function and a smooth measure on Y.

Unfortunately, we do not have an extension of this Theorem to the positive characteristic case. In fact it is not even clear how to formulate it correctly since there is no universally accepted definition of rational singularities (see [Har98, Smi97, Bha12, Kov00] for several related notions).

For this paper we choose the following notion of rational singularities in positive characteristic.

Definition 1.3.3. *Let $\mathbf{Z}$ be a variety defined over an arbitrary field. We say that the singularities of $\mathbf{Z}$ are rational if $\mathbf{Z}$ is Cohen-Macaulay, normal, and admits a resolution of singularities $\eta : \tilde{\mathbf{Z}} \to \mathbf{Z}$ such that the natural morphism $\eta_*(\Omega_{\tilde{\mathbf{Z}}}) \to i_*(\Omega_{\mathbf{Z}^{sm}})$ is an isomorphism. Here $i : \mathbf{Z}^{sm} \hookrightarrow \mathbf{Z}$ is the embedding of the smooth locus and Ω denotes the sheaf of top differential forms.*

Remark 1.3.4. *In characteristic zero, this notion is equivalent to rational singularities, see e.g.* [AA16, Appendix B, Proposition 6.2].

Next we give several extensions of the notion of FRS maps to positive characteristic:

Definition 1.3.5. *Let $\phi : \mathbf{M} \to \mathbf{N}$ be a flat morphism of smooth algebraic varieties over a local field F of arbitrary characteristic. Assume that the fibers of ϕ are reduced and normal.*

(1) We say that ϕ is **geometrically FRS** *(in short geo-FRS) if for any $y \in \mathbf{N}(\bar{F})$, the singularities of $\phi^{-1}(y)$ are rational.*

(2) We say that ϕ is **analytically FRS** *(in short an-FRS) if for any smooth compactly supported measure μ_M on $M := \mathbf{M}(F)$ there exist a smooth compactly supported measure μ_N on $N := \mathbf{N}(F)$, and a bounded function f on N such that*

$$\phi_*(\mu_M) = f\mu_N.$$

(3) We say that ϕ is **almost analytically FRS** *(in short almost an-FRS) if for any smooth compactly supported measure μ_M on M there exist smooth compactly supported measure μ_N and a function f on N such that*

$$\phi_*(\mu_M) = f\mu_N.$$

and $f \in L^r(N)$ for all $r \in [1, \infty)$

Using extension of scalars from $\mathbb{F}_\ell$ to $\mathbb{F}_\ell((t))$ we will apply these notions also for maps of varieties over $\mathbb{F}_\ell$.

Remark 1.3.6. *As in Remark 1.3.4, in characteristic zero, the geo-FRS property is equivalent to FRS property. Also, by Theorem 1.3.2, in this case each of them implies the an-FRS property.*

In this language, Theorem D implies that, under an appropriate assumption of existence of resolution of singularities, the Chevalley map, $p : \mathfrak{g} \to \mathfrak{c}$, is almost an-FRS. Similarly, the content of Theorem E is that, for $\mathrm{char}(\mathbb{F}_\ell) > \frac{n}{2}$ the Chevalley map, $p : \mathfrak{g} \to \mathfrak{c}$, is an-FRS.

The following is a positive characteristic analogue of Theorem 1.3.2:

Conjecture F. *Let $\phi : \mathbf{M} \to \mathbf{N}$ be a flat morphism of smooth algebraic varieties defined over a local field F whose fibers are reduced and normal. Assume that ϕ is geo-FRS. Then ϕ is an-FRS.*

We explain in §1.3.2 below that this would imply the following:

Conjecture G. *The Chevalley map $p : \underline{\mathfrak{g}} \to \underline{\mathfrak{c}}$ is an-FRS.*

Theorems D and E are partial results towards this conjecture.

1.3.2. *The Springer resolution.* The nilpotent cone has a natural resolution of singularities

$$T^*(\mathfrak{B}) \to \mathbf{N}$$

by the cotangent bundle to the flag variety, called the Springer resolution. In characteristic zero, one can use this resolution in order to prove that the singularities of the nilpotent cone are rational (see [Hes76, Theorem A]). We list now two important corollaries of this fact:

(I) The jet schemes of the nilpotent cone are irreducible of the expected dimensions (See [Mus01, Appendix]).

(II) The Chevalley map $p : \underline{\mathfrak{g}} \to \underline{\mathfrak{c}}$ is an-FRS. This follows from the rationality of the singularities of the nilpotent cone using [AA16, Theorem 3.4]. This fact is essentially equivalent to the bounds on orbital integrals established in [HC70, Theorem 13, page 68]. Note that [HC70, Theorem 13] is an important ingredient in Harish-Chandra's integrability theorem for characters of irreducible representations of reductive p-adic groups (See [HC99]). We will discuss this below in more details (see §1.3.3).

In positive characteristic, these results are not known. In more details, the Springer resolution exists in any characteristic, moreover, according to our definition of rational singularities (Definition 1.3.1) one can deduce from it that the singularities of the fibers of the Chevalley map are rational (it follows from the proof in [Hin91] of [Hes76, Theorem A]). So we have:

Proposition 1.3.7. *The Chevalley map $p : \underline{\mathfrak{g}} \to \underline{\mathfrak{c}}$ is geo-FRS.* [2]

However, the main results of [AA16, Mus01, HC70] are not known in positive characteristic for the following reasons:

[2] Indeed, by Jordan-Chevalley decomposition it is enough to show that the Springer resolution satisfy the condition of Definition 1.3.3. This is proven in [Hin91] for a generalization of the Springer resolution. The statements in [Hin91] are formulated for characteristic zero, but the proof of this statement is valid in arbitrary characteristic.

(1) [AA16, Mus01] use repeatedly the existence of resolution of singularities which is not known in positive characteristic.
(2) [AA16, Mus01] use the Grauert-Riemenschneider theorem, which is not valid (in general) over fields of positive characteristic (see [Ray78] and [MvdK92]).
(3) The proof of [HC70, Theorem 13] uses the fact that the Lie algebra of G is the direct sum of its center and its derived algebra. This is not true in positive characteristic.
(4) The proof of [HC70, Theorem 13] uses the Jordan-Chevalley decomposition. This decomposition does not exist as is over local fields of positive characteristic, since there are elements with irreducible totally inseparable characteristic polynomial.

We view Theorems A - E as partial analogs of (I) and (II), and as evidence for Conjectures F and G.

1.3.3. *Harish-Chandra's integrability theorem.* Another motivation for Theorems C and D is the following celebrated result by Harish-Chandra.

Theorem 1.3.8 ([HC99]). *Let F be a local non-Archimedean field with* **characteristic zero.** *Let $\mathbf{H}$ be a reductive group defined over F. Let $H = \mathbf{H}(F)$ and fix a Haar measure μ on H. Let (ρ, V) be an irreducible smooth representation of H. Consider the distributional character χ_ρ of ρ defined by $\chi_\rho(f) = trace(\rho(f\mu))$ where*

$$\rho(f\mu)(v) = \int_H f(g)\rho(g)v\mu$$

and $f \in C_c^\infty(G), v \in V$.

Then χ_ρ is represented by a locally integrable function on H.

An important step in the proof of Theorem 1.3.8 is a bound on orbital integrals [HC70, Theorem 13]. This bound is essentially equivalent to the fact that the Chevalley map is an-FRS. Thus we view Theorem D as a partial positive characteristic analogue of [HC70, Theorem 13].

In [AGKSb] we will use Theorems D and E in order to prove a positive characteristic version of Harish-Chandra local integrability theorem (Theorem 1.3.8) for irreducible cuspidal representations of GL_n under the assumption of the existence of resolutions of singularities as in Theorem D or the assumption on characteristic of Theorem E.

The main statement of [Lem96] is Theorem 1.3.8 for GL_n in positive characteristic. However, this proof contains a gap that we explain in details in [AGKSb]. Moreover, the statement of [Lem21, §5.3 Corollary 1] is equivalent to the statement of Theorem E but without limitation on characteristic. However, the proof of [Lem21, §5.3 Corollary 1] contains a gap. Namely, it is based on the lemma in [Lem21, §3.7] which is wrong as stated. This mistake is a propagation of an earlier mistake from [Lem97, Lemma 5.4.2], which in turn comes from a mistake in [Lem96, Lemma 2.3.2].

1.4. **Further related results.** One can use the methods of [CGH14] to prove that Theorems A, B, and D are valid for large enough characteristic of $\mathbb{F}_\ell$. However, no explicit bound on the characteristic can be obtained in this way.

1.5. **Ideas of the proofs.**

1.5.1. *The original Harish-Chandra's argument.* The starting point for this paper is the original Harish-Chandra's proof of the bound on the orbital integrals of a compactly supported function on $\mathfrak{gl}_n$ in the characteristic zero case (See [HC70, Theorem 13]). This can be reformulated as the statement that the Chevalley map is an-FRS. Let us briefly recall Harish-Chandra's argument in the language of an-FRS maps:

(1) Decompose $\mathfrak{gl}_n$ as $\mathfrak{sl}_n \oplus \mathfrak{gl}_1$ and reduce the statement to the statement that the Chevalley map $p' : \mathfrak{sl}_n \to \mathfrak{c}'$ is an-FRS.
(2) Deduce from the induction hypothesis that the map $p'|_{\mathfrak{sl}_n \smallsetminus N}$ is an-FRS, where N is the nilpotent cone.
(3) Prove by descending induction that $p'|_{\mathfrak{sl}_n \smallsetminus S}$ is an-FRS, where $S \subset N$ is a closed GL_n-invariant subset. The induction is on S:
    (a) For each nilpotent x use the Slodowy slice L_x at x to reduce the statement that $p'|_{L_x}$ is an-FRS to the statement that $p'|_{L_x \smallsetminus 0}$ is an-FRS.
    (b) Use the action of $\mathbb{G}_m$ on L_x and an estimate on its eigenvalues in order to prove that $p'|_{L_x}$ is an-FRS assuming the fact that $p'|_{L_x \smallsetminus 0}$ is an-FRS.

When going to positive characteristic this argument has several problems:

- Step (1) is invalid.
- Step (2) is invalid as stated, but one can adapt it to prove that $p|_{\mathfrak{sl}_n \smallsetminus N_{insep}}$ is an-FRS where N_{insep} is the collection of matrices with purely inseparable characteristic polynomial.[3]
- Step (3) is invalid as stated, but one can replace the Slodowy slice ([Slo80]) with other constructions (see §6.2). However, this method can be applied directly only to the nilpotent cone N and not to N_{insep}.

1.5.2. *An-FRS over the origin.* One can start with proving a weaker statement. Namely, that p is an-FRS over the origin. This means that for every smooth compactly supported measure μ, the density of $p_*(\mu)$ at $0 \in \mathfrak{c}$ is finite. Note that this does not imply that this density is bounded in any neighborhood of the origin. In fact, in order to enable Harish-Chandra's argument, we need a very narrow definition of density at a point - the limit of the average density for a very specific sequence of balls that converges to 0.

[3]For example the matrix $\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$ over the field $\mathbb{F}_2((t))$.

For the weaker statement, steps (1) and (2) become obsolete, since it is obvious that $p_{\mathfrak{g}\smallsetminus N}$ is an-FRS over the origin (as the origin is not in its image). One can adapt step (3) to work in this case.

1.5.3. *Effectively an-FRS over the origin.* The property of being an-FRS over the origin by itself is not useful for us, since we can not use it in order to deduce any information outside the origin. In order to make it useful we have to consider a version which is uniform over finite extensions of $\mathbb{F}_\ell$. This leads us to the notion of effectively an-FRS over the origin (see Definition 6.0.2 below). Our proofs of Theorems C and D are based on the proof that p is effectively an-FRS over the origin (see Theorem 6.0.4 below). The proof follows the lines described above, but with several important adaptations:

- One has to properly define a setting where one can state effective results that are uniform on finite extensions of $\mathbb{F}_\ell$. This we did in [AGKSa].
- One has to reprove standard facts from differential geometry (such as the implicit function theorem) in this effective setting. This was also done in [AGKSa].
- One has to carefully define the notion of an-FRS maps in a way that makes it amenable for Harsh-Chandra's argument. The key points here are to work with a specific sequence of ellipsoids around the origin and to require the bound on the average density to be uniform over the entire sequence (even for large ellipsoids). This makes the fact that $p|_{\mathfrak{gl}_n\smallsetminus N}$ is effectively an-FRS over the origin not obvious, but still correct (see Lemma 6.1.3 below).
- The Harish-Chandra's argument uses (implicitly) the fact that the notion of an-FRS map is local with respect to smooth covers which are surjective on the level of F points. This locality is not clear for maps that are effectively an-FRS over the origin. However, we show that this notion is local with respect to a class of smooth covers which we call effectively surjective (see Definition 4.1.5 below). In [AGKSa] we proved that this class of covers includes the Nisnevich covers (See Proposition 4.2.7 below). This allows us to adapt Harish-Chandra's argument.

1.5.4. *Proof of Theorems A, B, and C.* We deduce Theorem A from the fact that p is effectively an-FRS over the origin using the Lang-Weil bounds. If the ellipsoids in the definition of effectively an-FRS over the origin would be balls, this would be straightforward - the average density would be exactly the limit of the normalized number of points in the jet-scheme. In our case, the bounds on the average density provide bounds on the dimension of some weighted versions of the jets schemes. We deduce the desired bounds on dimensions from this bound using the semi-continuity of dimension of fibers. This is done in §7 (Theorem A itself is proven in §8 but the actual argument is in §7.)

We deduce Theorem B from Theorem A using semi-continuity of dimension again. Theorem C follows easily from Theorem B. See §8.

1.5.5. *Proof of Theorem D.* We now want to go back to an-FRS property but this time over the entire range. This we do only under the additional assumption of existence of resolution of singularities.

For a variety $\mathbf{X}$ (equipped with a top form on its smooth locus) we can consider the following quantity: the volume of a ball of radius R in $\mathbf{X}^{sm}(\mathbb{F}_{\ell^k}((t)))$. Here $\mathbf{X}^{sm}$ denotes the smooth locus of $\mathbf{X}$. Note that the notion of ball in $\mathbf{X}^{sm}$ is defined in such a way that the singular locus of $\mathbf{X}$ is infinitely far. We can study the asymptotics of this volume both as a function of R and as a function of k. We note that our analysis is local on $\mathbf{X}$ so all the balls are intersected with a fixed ball in $\mathbf{X}$.

Under our assumption of existence of resolution of singularities we relate these 2 asymptotics. See Theorem 9.0.3.

We apply Theorem 9.0.3 to the variety $\underline{\mathfrak{g}}_i$. We relate the asymptotic behavior of the above volume with respect to k to the number of points on the jet schemes of $\underline{\mathfrak{g}}_i$. See Theorem 10.0.1. We use Theorem 9.0.3 to deduce bounds on the asymptotics of the above volume with respect to R.

Now we wish to relate this asymptotics to the desired L^i property. We fix a measure μ on $\underline{\mathfrak{g}}(\mathbb{F}_\ell((t)))$ and consider its pushforward under the Chevalley map $p : \underline{\mathfrak{g}}(\mathbb{F}_\ell((t))) \to \underline{\mathfrak{c}}(\mathbb{F}_\ell((t)))$. Denote this resulting measure by ν. We consider the L^i norm of ν outside an ε-neighborhood of the singular locus of p.

We relate the asymptotics of this L^i norm w.r.t. ε to the above asymptotics w.r.t. R. We use a standard analytic argument to deduce from this a bound on the $L^{i'}$-norm for any $i' < i$. See §11.

1.5.6. *Proof of Theorem E.* We follow the original Harish-Chandra's argument. The first step is to analyze the situation near (non-central) semisimple elements, using the induction hypothesis.

In the characteristic zero situation this proves the result outside the Minkowski sum of the nilpotent cone and the set of scalar matrices.

In the positive characteristic situation this proves the result outside the cone of matrices whose characteristic polynomial is purely inseparable.

In general, this cone is rather complicated, however, under our assumption on the characteristic, only its regular part exhibits this complexity. Since the statement is obvious for regular matrices, we can ignore this complexity, and again deduce the result outside the Minkowski sum of the (non-regular) nilpotent cone and the set of scalar matrices.

This still does not finish the problem, since we do not have the splitting of $\underline{\mathfrak{g}}$ to scalar and traceless matrices. So we use our assumption on the characteristic again in order to choose an analog of the Slodowy slice (to a non-regular orbit) which will have such a splitting. See Lemma 13.3.1.

Though the slice has a splitting, $\mathfrak{c}$ does not admit a corresponding splitting. Thus the original Harish-Chandra's homogeneity argument does not work here. One has to tweak it in order to take into account the one-dimensional center (see Lemma 13.4.4). This makes it less sharp, so it fails to give the an-FRS property around the subregular orbit (though it still gives the almost an-FRS property). Therefore we prove the an-FRS property around the subregular orbit using a direct computation - see Step 4 of the proof of Theorem E.

1.5.7. *The role of the assumption* $\mathbf{G} = \mathrm{GL}_n$. We used the assumption $\mathbf{G} = \mathrm{GL}_n$ in order to make all explicit computations easier. However, our argument does not use any statement that inherently depends on this assumption (such as existence of mirabolic subgroup, stability of adjoint orbits, or the Richardson property of all nilpotent orbits).

Moreover, for non-type A groups in good characteristic the analog of N_{insep} coincides with the nilpotent cone N. Therefore, we expect the original Harish-Chandra's argument to allow reduction to type A. Hence, we expect that in good characteristic[4], the conclusion of Theorem D for general reductive group will only require the assumptions on resolution of the variety $\underline{\mathfrak{g}}_i$ for $\underline{\mathfrak{g}} = \mathfrak{gl}_n$.

1.6. **Structure of the paper.** In §2 we fix some conventions and recall some standard facts.

In §3 we give a short overview of the theory of norms on algebraic varieties over local fields developed in [Kot05, §18].

In §4, we recall the theory of rectified algebraic varieties and balls and measures on them, introduced in [AGKSa].

We recall the main results of [AGKSa], which are uniform analogs of standard results from local differential topology, including the implicit function theorem and study the behavior of smooth measures under push forward with respect to submersions.

We also introduce the notion of effectively surjective map from [AGKSa], which is a surjective map such that we can control the norm of a preimage in terms of the norm of the point in the target. We recall a statement from [AGKSa] that implies that any Nisnevich cover is effectively surjective.

In §5 we recall some standard facts on the Chevalley map that are less standard in positive characteristic.

In §6 we introduce the notion of effectively an-FRS over the origin and prove that the Chevalley map has this property.

In §7 we prove a bound on the dimension of the jets of the fiber of an effectively an-FRS map over the origin.

In §8 we deduce Theorems A, B and C.

In §9 we introduce several notions related to the asymptotics of the volumes of balls in the smooth loci of varieties over local function fields. We call these notions analytic, geometric and asymptotic almost integrability.

We show that all these notions are equivalent under the assumption of existence of an appropriate resolution of singularities.

[4]see e.g. [SS70, I, §4] for the definition of this notion

In §10 we prove that the varieties $\underline{\mathfrak{g}}_i$ are asymptotically almost integrable.
In §11 we prove Theorem D.
In §12 we give several variations of Theorem D.
In §13 we prove Theorem E.

1.7. **Acknowledgments.** We would like to thank Dan Abramovich and Michael Temkin for enlightening conversations about resolution of singularities. We would also like to thank Nir Avni for many conversations on algebro geometric analysis.

During the preparation of this paper, A.A., D.G. and E.S. were partially supported by the ISF grant no. 1781/23. D.K. was partially supported by an ERC grant 101142781.

2. Notations and Preliminaries

2.1. **Conventions.**

(1) By a variety we mean a reduced scheme of finite type over a field. Unless stated otherwise this field will be $\mathbb{F}_\ell$.
(2) When we consider a fiber product of varieties, and fibers of maps between varieties, we always consider it in the category of schemes.
(3) We will describe subschemes and morphisms of varieties and schemes using set-theoretical language, when no ambiguity is possible.
(4) We will usually denote algebraic varieties by bold face letters (such as $\mathbf{X}$).
(5) For Gothic letters we use underline instead of boldface.
(6) We will use the same letter to denote a morphism between algebraic varieties and the corresponding map between the sets of their F-points for various fields F.
(7) We will use the symbol $\square$ in the middle of a square diagram in order to denote that a square is Cartesian.
(8) A big open set of an algebraic variety $\mathbf{Z}$ is an open set whose complement is of co-dimension at least 2 (in each component)
(9) For an algebraic variety $\mathbf{X}$, we denote by $\mathbf{X}^{sm}$ the variety of smooth points of $\mathbf{X}$. We also denote by $\mathbf{X}^{sing}$ the variety of singular points.
(10) We will abbreviate SNC divisor for strict normal crossings divisor.
(11) By a strong resolution of singularities of a variety $\mathbf{X}$ we mean a proper birational map $\phi : \tilde{\mathbf{X}} \to \mathbf{X}$ s.t. $\tilde{\mathbf{X}}$ is smooth, ϕ is an isomorphism over $\mathbf{X}^{sm}$ and the inverse image of $\mathbf{X}^{sing}$ in $\tilde{\mathbf{X}}$, considered as a variety, is an SNC divisor.
(12) If F is a local field and X is an analytic manifold over F (In the sense of [Ser06, Part II, Chapter 3]) and $r > 1$ we denote by $L^r_{\text{loc}}(X)$ the space of functions on X which are locally in L^r. That is, functions f s.t. for any open analytic embedding $\phi : U \to X$, with $U \subset F^M$ precompact, we have $f \circ \phi \in L^r_{\text{loc}}(U)$. We also define
$$L^{<\infty}_{\text{loc}}(X) = \bigcup_r L^r_{\text{loc}}(X).$$

(13) When considering elements in these function spaces, we will not distinguish between functions on X and functions defined almost everywhere in X.

2.2. **Forms and measures.**

Definition 2.2.1. *For a top form ω on a smooth algebraic variety $\mathbf{X}$ defined over a local field F, we denote the corresponding measure on $X := \mathbf{X}(F)$ by $|\omega|$.*

Notation 2.2.2. *For a smooth morphism $\gamma : \mathbf{Z}_1 \to \mathbf{Z}_2$, a top differential form $\omega_{\mathbf{Z}_2}$ on $\mathbf{Z}_2$, and a relative top differential form ω_γ on $\mathbf{Z}_1$ with respect to γ, denote the corresponding top differential form on $\mathbf{Z}_1$ by $\omega_{\mathbf{Z}_2} * \omega_\gamma$.*

We use the same notation for rational top-forms. Also in this case, we do not have to require that $\mathbf{Z}_i$ and γ are smooth, instead it is enough to require that γ is generically smooth.

Definition 2.2.3. *Given a Cartesian square of smooth morphisms and smooth varieties:*

$$\begin{array}{ccc} \mathbf{V} & \longrightarrow & \mathbf{Z}_1 \\ \downarrow & \square & \downarrow \\ \mathbf{Z}_2 & \longrightarrow & \mathbf{Z} \end{array}$$

and top forms ω, ω_i on $\mathbf{Z}, \mathbf{Z}_i$ define a form $\omega_1 \times_\omega \omega_2$ on $\mathbf{V}$ in the following way:

- *Let ω_i' be a Gelfand-Leray relative form on $\mathbf{Z_i}$ w.r.t. the map $\mathbf{Z}_i \to \mathbf{Z}$.*
- *$\omega_1' \boxtimes_{\mathbf{Z}} \omega_2'$ is the corresponding relative form on $\mathbf{V}$ w.r.t. the map $\gamma : \mathbf{V} \to \mathbf{Z}$.*
- *$\omega_1 \times_\omega \omega_2 := \omega * (\omega_1' \boxtimes_{\mathbf{Z}} \omega_2')$.*

We use the same notation for rational top-forms. Also in this case, we do not have to require that $\mathbf{Z}_i$, $\mathbf{Z}$ and γ be smooth, instead it is enough to require that they are generically smooth.

3. Norms

In this section, we recall basic parts of the theory of norms developed in [Kot05, §18], and prove an integrability result about this theory (See Proposition 3.0.1 below). We will use the following notions from [Kot05, §18].

(1) An abstract norm on a set Z is a positive real-valued function $||\cdot||_Z$ on Z such that $||x||_Z \geq 1$ for all $x \in Z$.
(2) For two abstract norms $||\cdot||_Z^1, ||\cdot||_Z^2$ on Z we say that $||x||_Z^1 \prec ||x||_Z^2$ if there is a constant $c > 1$ s.t. $||x||_Z^1 < c(||x||_Z^2)^c$.
(3) We say that two abstract norms $||\cdot||_Z^1, ||\cdot||_Z^2$ on Z are equivalent, and denote this as $||\cdot||_Z^1 \sim ||\cdot||_Z^2$, if $||x||_Z^1 \prec ||x||_Z^2 \prec ||x||_Z^1$.

(4) Let $\mathbf{M}$ be an algebraic variety defined over a local field F. In [Kot05, §18] there is a definition of a canonical equivalence class of abstract norms in $M = \mathbf{M}(F)$. The abstract norms in this class are called norms on M.

The main result of this section is the following:

Proposition 3.0.1. *Let $\mathbf{U} \subset \mathbf{X}$ be an open dense subset of a smooth variety. Let $X = \mathbf{X}(\mathbb{F}_\ell((t)))$ and $U = \mathbf{U}(\mathbb{F}_\ell((t)))$. Let $||\cdot||_U$ be a norm on U. Then*

$$\log_\ell \circ ||\cdot||_U \in L^{<\infty}_{\mathrm{loc}}(X).$$

For the proof we will need some preparations.

Lemma 3.0.2 (cf. [GH25, Theorem 1.3])**.** *Let $\mathbf{X}$ be a smooth algebraic variety defined over a local field F. Let $f \in \mathcal{O}_{\mathbf{X}}(\mathbf{X})$ be a non-zero divisor. Let $X = \mathbf{X}(F)$. Then there exist $\varepsilon > 0$ s.t. $|f|^{-\varepsilon} \in L^1_{\mathrm{loc}}(X)$.*

Corollary 3.0.3. *Let $\mathbf{X}$ be a smooth algebraic variety defined over a local field F. Let $f \in \mathcal{O}_{\mathbf{X}}(\mathbf{X})$. Let $X = \mathbf{X}(F)$. Then* $\log(|f|) \in L^{<\infty}_{\mathrm{loc}}(X)$.

Proof. WLOG assume that we have an invertible to form ω on $\mathbf{X}$. Fix a compact $C \subset X$. Let

$$C_i := \{x \in C : i \leq |f(x)|^{-1} < i + 1\}$$

Let $m_i := |\omega|(C_i)$. Let ε be as in the lemma. Then we have

$$\sum_{i=1}^{\infty} i^{\varepsilon} m_i < \infty.$$

So, for any $k > 0$ we obtain

$$\sum_{i=1}^{\infty} (\log(i+1))^k m_i < \infty.$$

This implies the assertion. ∎

Proof of Proposition 3.0.1. Note that this statement does not depend on the norm so we will choose the norm at our convenience.

Case 1. $\mathbf{X}$ is affine and $\mathbf{U} = \mathbf{X}_f \subset \mathbf{X}$ is principal open affine subset:
Take any norm $||\cdot||_{\mathbf{X}}$ on X and take

$$||x||_{\mathbf{U}} := \max(||x||_{\mathbf{X}}, |f(x)|^{-1})$$

The assertion follows now from Corollary 3.0.3.

Case 2. $\mathbf{X}$ is affine:
Follows from the previous case.

Case 3. $\mathbf{X}$ is general case:
Follows from the previous case.

∎

4. Rectified algebraic varieties

In this section, we recall the theory of rectified algebraic varieties and balls and measures on them, introduced in [AGKSa]. This is a framework for quantitative statements on distances and measures when studying algebraic varieties and morphisms of algebraic varieties over local fields of the type $\mathbb{F}_\ell((t))$. It allows to formulate uniform statements with respect to finite extensions of $\mathbb{F}_\ell$.

We recall the main results of [AGKSa] which are uniform analogues of standard results from local differential topology, including the implicit function theorem and study the behavior of smooth measures under push forward with respect to submersions.

Part of this theory is analogous to the theory of norms recalled above.

We also introduce the notion of effectively surjective map from [AGKSa], which is a surjective map such that we can control norm of a preimage in terms of the norm of the point in the target. We recall a statement from [AGKSa] that implies that any Nisnevich cover is effectively surjective. See Proposition 4.2.7.

4.1. **Notions.** We recall here the main notions from [AGKSa]:

Definition 4.1.1. *Let $\mathbf{X}$ be a smooth algebraic variety over $\mathbb{F}_\ell$.*

1. *A rectification of $\mathbf{X}$ is a finite open cover $\mathbf{X} \subset \bigcup_{\alpha \in I} \mathbf{U}_\alpha$ with closed embeddings $i_\alpha : \mathbf{U}_\alpha \to \mathbb{A}^M$.*
2. *We will call a rectification simple if $|I| = 1$.*
3. *By a rectified variety we will mean a smooth algebraic variety over $\mathbb{F}_\ell$ equipped with a rectification. By a map or a morphism of such we just mean a morphism of the underlying algebraic varieties.*
4. *A μ-rectification of $\mathbf{X}$ is a rectification of $\mathbf{X}$ together with invertible top differential forms $\omega_\alpha \in \Omega^{top}(\mathbf{U}_\alpha)$.*
5. *We define similarly the notion of a μ-rectified variety, and simple μ-rectification.*

Definition 4.1.2.

1. *Let $(\mathbf{X}, \mathbf{U}_\alpha, i_\alpha)$ be a rectified variety. Then, for any $k \in \mathbb{N}$ and $m \in \mathbb{Z}$ define:*
   - *(a) $B_m^{\mathbf{X},k} := \bigcup_\alpha i_\alpha^{-1}\left(t^{-m}\mathbb{F}_{\ell^k}[[t]]^M\right)$.*
   - *(b) $B_\infty^{\mathbf{X},k} := \bigcup_{m \in \mathbb{N}} B_m^{\mathbf{X},k} = \mathbf{X}(\mathbb{F}_{\ell^k}((t)))$.*
   - *(c) For $x \in \mathbf{X}(\mathbb{F}_{\ell^k}((t)))$ define a ball around x*
   $$B_m^{\mathbf{X},k}(x) := \bigcup_{\alpha \text{ s.t. } x \in U_\alpha(F_{\ell^k}((t)))} i_\alpha^{-1}\left(i_\alpha(x) + t^{-m}\mathbb{F}_{\ell^k}[[t]]^M\right).$$
   - *(d) For $\mathbf{Z} \subset \mathbf{X}$ define*
   $$B_m^{\mathbf{X},k}(\mathbf{Z}) := \bigcup_{z \in \mathbf{Z}(\mathbb{F}_{\ell^k}((t)))} B_m^{\mathbf{X},k}(z).$$

(2) *Let* $(\mathbf{X}, \mathbf{U}_\alpha, i_\alpha, \omega_\alpha)$ *be a* μ*-rectified variety. Then, for any positive integers* k, m *define a measure on* $\mathbf{X}(\mathbb{F}_{\ell^k}((t)))$ *supported on* $B_m^{\mathbf{X},k}$ *defined by*

$$\mu_m^{\mathbf{X},k} := \sum_\alpha |(\omega_\alpha)_{\mathbb{F}_{\ell^k}((t))}| \cdot 1_{i_\alpha^{-1}\left(t^{-m}\mathbb{F}_{\ell^k}[[t]]^M\right)}.$$

(3) *If* $\mathbf{X}$ *is an affine space, we denote by* $\mu^{\mathbf{X},k}$ *the Haar measure on* $\mathbf{X}(\mathbb{F}_{\ell^k}((t)))$ *normalized s.t.* $\mu^{\mathbf{X},k}(\mathbf{X}(\mathbb{F}_{\ell^k}[[t]]) = 1$.

Definition 4.1.3. *By an almost affine space we mean a principal open subset in an affine space defined over* $\mathbb{F}_\ell$. *Note that any almost affine space is equipped with a natural simple (*μ*-)rectification that we will call the standard (*μ*-)rectification on this space.*

When dealing with such space, if we are not fixing a μ*-rectification on it, the above notions of balls and measures will refer to the standard rectification.*

Definition 4.1.4. *Let* $\mathbf{X}$ *be a rectified variety. Let* $m, k \in \mathbb{N}$. *We say that* $f \in C^\infty(B_\infty^{\mathbf{X},k})$ *is* m*-smooth if for any* $x \in B_\infty^{\mathbf{X},k}$ *the function* $f|_{B_{-m}^{\mathbf{X},k}(x)}$ *is constant.*

Definition 4.1.5. *Let* $\gamma : \mathbf{X} \to \mathbf{Y}$ *be a map of rectified varieties. We say that* γ *is effectively surjective iff for any* $m \in \mathbb{N}$ *there is* $m' \in \mathbb{N}$ *s.t. for every* $k \in \mathbb{N}$ *we have*

$$\gamma(B_{m'}^{\mathbf{X},k}) \supset B_m^{\mathbf{Y},k}.$$

4.2. **Statements.** We recall here the main statements of [AGKSa].

The following is obvious:

Lemma 4.2.1 ([AGKSa, Lemma 3.4])**.** *Let* $\mathbf{X}$ *be a rectified variety. Then for any 2 integers* $m_1, m_2 \in \mathbb{N}$ *we have:*

(1) *If* $x \in B_{m_2}^{\mathbf{X},k}$ *then* $B_{-m_1}^{\mathbf{X},k}(x) \subset B_{m_2}^{\mathbf{X},k}$.
(2) *If* $x \in B_\infty^{\mathbf{X},k}$ *and* $y \in B_{-m_1}^{\mathbf{X},k}(x)$ *then* $x \in B_{-m_2}^{\mathbf{X},k}(y)$.
(3) *If the rectification of* $\mathbf{X}$ *is simple and* $m_1 \geq m_2$, *then*

$$B_{-m_1}^{\mathbf{X},k}(x) \subset B_{-m_2}^{\mathbf{X},k}(x) = B_{-m_2}^{\mathbf{X},k}(y)$$

for any $x \in \mathbf{X}(\mathbb{F}_{\ell^k}((t)))$ *and* $y \in B_{-m_2}^{\mathbf{X},k}(x)$.

The following lemma says that regular maps are uniformly continuous and bounded on balls, in a way which is also uniform on the residue field:

Lemma 4.2.2 ([AGKSa, Proposition 3.5])**.** *Let* $\gamma : \mathbf{X} \to \mathbf{Y}$ *be a map of rectified algebraic varieties. Then for any* $m \in \mathbb{N}$ *there is* $m' > m$ *s.t. for any* k *and any* $x \in B_m^{\mathbf{X},k}$ *we have*

(i) $\gamma(B_m^{\mathbf{X},k}) \subset B_{m'}^{\mathbf{Y},k}$.
(ii) $\gamma(B_{-m'}^{\mathbf{X},k}(x)) \subset B_{-m}^{\mathbf{Y},k}(\gamma(x))$.

The following two corollaries imply that all the statements on balls that we formulate do not depend on the rectification.

Corollary 4.2.3 ([AGKSa, Corollary 3.6]). *Let $\mathbf{X}_1, \mathbf{X}_2$ be two copies of the same $\mathbb{F}_\ell$-variety with two (possibly different) rectifications. Let $\mathbf{Z} \subset \mathbf{X}_1$ be a closed subvariety. Then*

(1) for any $m \in \mathbb{N}$ there is $m' \in \mathbb{N}$ s.t. for any $k \in \mathbb{N}$ we have:
- *(a) $B_m^{\mathbf{X}_1,k} \subset B_{m'}^{\mathbf{X}_2,k}$.*
- *(b) for any $x \in \mathbf{X}_1(\mathbb{F}_{\ell^k}((t)))$ we have $B_m^{\mathbf{X}_1,k}(x) \subset B_{m'}^{\mathbf{X}_2,k}(x)$.*
- *(c) $B_m^{\mathbf{X}_1,k}(\mathbf{Z}) \subset B_{m'}^{\mathbf{X}_2,k}(\mathbf{Z})$.*

(2) For any μ-rectifications of $\mathbf{X}_i$ and $m \in \mathbb{N}$, there exists m' s.t. for any k we have:

$$\mu_m^{\mathbf{X}_1,k} < \ell^{km'} \mu_{m'}^{\mathbf{X}_2,k}$$

Corollary 4.2.4 ([AGKSa, Lemma 6.2]). *The property of a map $\gamma : \mathbf{X} \to \mathbf{Y}$ being effectively surjective does not depend on the rectifications on the varieties $\mathbf{X}$ and $\mathbf{Y}$.*

At some point we will need the following stronger version of Corollary 4.2.3 (1a):

Lemma 4.2.5 ([AGKSa, Lemma 3.7]). *Let $\mathbf{X}_1, \mathbf{X}_2$ be two copies of the same $\mathbb{F}_\ell$-variety with two (possibly different) rectifications. Then there exists $a \in \mathbb{N}$ s.t. for any $m, k \in \mathbb{N}$ we have:*

$$B_m^{\mathbf{X}_1,k} \subset B_{am+a}^{\mathbf{X}_2,k}.$$

The next result is an effective version of the open mapping theorem:

Lemma 4.2.6 ([AGKSa, Theorem 4.2]). *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a smooth map of smooth (rectified) algebraic varieties. Then for any m there is m' s.t. for any k and any $x \in B_m^{\mathbf{X},k}$ we have*

$$\gamma(B_{-m}^{\mathbf{X},k}(x)) \supset B_{-m'}^{\mathbf{Y},k}(\gamma(x)).$$

The following is a criterion for effective subjectivity.

Proposition 4.2.7 ([AGKSa, Theorem 6.3]). *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a smooth map of algebraic varieties that is onto on the level of points for any field. Then γ is effectively surjective.*

The following four statements describe the behavior of the measures defined in §4.1 under push forward by a submersion. In particular they imply that all the statements that we formulate on these measures do not depend on the rectification.

Lemma 4.2.8 ([AGKSa, Lemma 3.9]). *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a submersion of μ-rectified varieties. Then for any m there is m' s.t.*

$$\gamma_*(\mu_m^{\mathbf{X},k}) < \ell^{km'} \mu_{m'}^{\mathbf{Y},k}.$$

Corollary 4.2.9 ([AGKSa, Corollary 3.10]). *Let $\mathbf{X}$ be a μ-rectified variety. Then for any $m \in \mathbb{N}$ there exists $M \in \mathbb{N}$ s.t. for any $k \in \mathbb{N}$:*

$$\mu_m^{\mathbf{X},k}(B_\infty^{\mathbf{X},k}) < \ell^{kM}.$$

Lemma 4.2.10 ([AGKSa, Corollary 6.8])**.** *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a submersion of μ-rectified varieties. Assume that γ is effectively surjective. Then for any $m \in \mathbb{N}$ there is $m' \in \mathbb{N}$ s.t. for any $k \in \mathbb{N}$ we have*

$$\mu_m^{\mathbf{Y},k} < \ell^{km'} \gamma_*(\mu_{m'}^{\mathbf{X},k})$$

Lemma 4.2.11 ([AGKSa, Theorem 5.7])**.** *Let $\gamma : \mathbf{X}_1 \to \mathbf{X}_2$ be a smooth map of μ-rectified varieties. Then for any $m \in \mathbb{N}$ there is $m' \in \mathbb{N}$ s.t. for any $k \in \mathbb{N}$ and any m-smooth function $g \in C_c^\infty(B_\infty^{\mathbf{X}_1,k})$ there is an m'-smooth function $f \in C_c^\infty(B_{m'}^{\mathbf{X}_2,k})$ s.t.:*

$$\gamma_*(g\mu_m^{\mathbf{X}_1,k}) = f \cdot \mu_{m'}^{\mathbf{X}_2,k}.$$

5. Basic geometry of the Chevalley map p

We recall that $p : \underline{\mathfrak{g}} \to \underline{\mathfrak{c}}$, is the Chevalley map sending a matrix to its characteristic polynomial. We need further notation:

Notation 5.0.1.

(1) Let $\underline{\mathfrak{t}}$ be the standard Cartan subalgebra of $\underline{\mathfrak{g}}$, consisting of diagonal matrices.
(2) Let $W := S_n$ be the Weyl group of GL_n.
(3) Identify $\mathfrak{c} \cong \mathfrak{t}//W$ by the Chevalley restriction theorem (see e.g. [Hum78, §23]). We will also identify it with the affine space $\mathbb{A}^n$.
(4) Denote by $q : \underline{\mathfrak{t}} \to \underline{\mathfrak{c}}$ the quotient map.
(5) Denote by $\underline{\mathfrak{g}}^{rss}$ the locus of regular semi-simple elements (i.e. matrices with n different eigenvalues over the algebraic closure).
(6) Denote $\underline{\mathfrak{c}}^{rss} := p(\underline{\mathfrak{g}}^{rss})$ the locus of separable polynomials in $\underline{\mathfrak{c}}$.

The following lemma follows immediately from miracle flatness (see [Sta25, Lemma 00R4]):

Lemma 5.0.2. *$q : \underline{\mathfrak{t}} \to \underline{\mathfrak{c}}$ is flat.*

Notation 5.0.3. *Let I be a finite set. For $w \in \mathbb{Z}^I$ define the w-degree of a monomial $\prod_{i\in I} x_i^{a_i}$ to be $\sum_{i \in I} a_i w(i)$. For a polynomial $f \in k[\mathbb{A}^I]$, we denote by $\sigma_w(f)$ the sum of the monomials of f with the highest w-degree.*

Proposition 5.0.4. *Let I, J be finite sets and let $\psi = (\psi_j)_{j\in J} : \mathbb{A}^I \to \mathbb{A}^J$ be a morphism such that $\psi(0) = 0$, fix $w \in \mathbb{Z}^I$. Let $\sigma_w(\psi) = (\sigma_w(\psi_j))_{j\in J}$. If $\sigma_w(\psi)$ is flat at 0, then so is ψ.*

Proof. The proof is identical to the proof of [AA16, Proposition 2.1.16], replacing [AA16, Proposition 2.1.1] with [DG67, IV, 11.3.10]. □

Corollary 5.0.5. *$p : \underline{\mathfrak{g}} \to \underline{\mathfrak{c}}$ is flat.*

Proof. Let $I = \{1, \dots, n\} \times \{1, \dots, n\}$ be the set of indices of $n \times n$ matrices. Following [BL96] we let $w : I \to \mathbb{Z}$ defined by $w(i,j) = \delta_{i,j}$. Identify $\underline{\mathfrak{g}}$ with $\mathbb{A}^I$. Using Proposition 5.0.4 and Lemma 5.0.2 we obtain that p is flat at 0. The assertion follows, using the homothety action of $\mathbb{G}_m$ on $\underline{\mathfrak{g}}$. □

Lemma 5.0.6. *The fibers of p are irreducible.*

Proof. This follows from the Jordan decomposition. $\square$

Notation 5.0.7. *Denote by $\underline{\mathfrak{g}}^r$ the smooth locus of p.*

Lemma 5.0.8. *$p|_{\underline{\mathfrak{g}}^r} : \underline{\mathfrak{g}}^r \to \underline{\mathfrak{c}}$ is onto.*

Proof. The companion matrix $C(f)$ attached to a polynomial $f \in \underline{\mathfrak{c}}$ is a regular matrix with characteristic polynomial equal to f. This proves the assertion. $\square$

Corollary 5.0.9. *The fibers of p are absolutely reduced. Furthermore, $\underline{\mathfrak{g}}^r$ is big in $\underline{\mathfrak{g}}$.*

Proof. By Lemma 5.0.8, each fiber of p has a generically reduced component. Hence by Lemma 5.0.6, the fibers of p are generically reduced. Since p is flat, its fibers are complete intersections. Thus by [Eis95, Exercise 18.9], they are reduced.

To show that $\underline{\mathfrak{g}}^r$ is big in $\underline{\mathfrak{g}}$ we let $\mathbf{Z}$ be its complement inside $\underline{\mathfrak{g}}$. By the above, for any $c \in \underline{\mathfrak{c}}$ we have $\dim(p^{-1}(c) \cap \mathbf{Z}) < \dim(\mathbf{Z})$. We obtain:

$$\dim(\mathbf{Z}) \leq \dim(\underline{\mathfrak{c}} - \underline{\mathfrak{c}}^{rss}) + \max_{c \in \underline{\mathfrak{c}} - \underline{\mathfrak{c}}^{rss}} \dim(p^{-1}(c) \cap \mathbf{Z}) \leq$$
$$\leq \dim(\underline{\mathfrak{c}}) - 1 + \max_{c \in \underline{\mathfrak{c}} - \underline{\mathfrak{c}}^{rss}} (\dim p^{-1}(c)) - 1 = \dim(\underline{\mathfrak{g}}) - 2$$

$\square$

Recall that for an integer $i \in \mathbb{N}$ we denote by $\underline{\mathfrak{g}}_i := \underline{\mathfrak{g}}^{\times_{\underline{\mathfrak{c}}} i}$ the i-folded fiber product of $\underline{\mathfrak{g}}$ with itself over $\underline{\mathfrak{c}}$ with respect to the map p. A-priory this is a scheme, but we can now show that it is reduced, so it is a variety.

Lemma 5.0.10. *The scheme $\underline{\mathfrak{g}}_i$ is reduced.*

Proof. By Corollary 5.0.9 the fibers of p are reduced. Therefore, so are the fibers of $\underline{\mathfrak{g}}_i \to \underline{\mathfrak{c}}$. This implies the assertion. $\square$

6. Effectively an-FRS over the origin

In this section we introduce a class of maps of algebraic variety to a vector space that refines the notion of an-FRS maps (See Definition 6.0.2).

The definition of this property is designed in a way that adapts the original Harish-Chandra's argument for an-FRS property of the Chevalley map to give some result also in positive characteristic. This enables us to prove this property for the Chevalley map (see Theorem 6.0.4 below). We later use this property in order to prove Theorem A.

Definition 6.0.1. *We say that an action of $\mathbb{G}_m$ on an affine space is positive if, in an appropriate coordinate system, it is given by*

$$\lambda \cdot (x_1, \dots, x_M) = (\lambda^{a_1} x_1, \dots, \lambda^{a_M} x_M),$$

with all a_i positive.

Definition 6.0.2 (effectively an-FRS over the origin)**.** *Let* $\mathbf{X}$ *be a rectified variety and let* $\gamma : \mathbf{X} \to \mathbf{Y}$ *be an algebraic map to an affine space* $\mathbf{Y}$*. Let* $\mathbb{G}_m$ *act positively on* $\mathbf{Y}$*. Choose the standard rectification of* $\mathbf{Y}$*.*

We say that γ *is effectively an-FRS over the origin if for any* $m \in \mathbb{N}$ *there exist* M *such that for any* $k \in \mathbb{N}$ *and* $a \in \mathbb{Z}$ *we have*

$$\frac{\left\langle \gamma_* \left(\mu_m^{\mathbf{X},k}\right), 1_{t^a \cdot B_0^{\mathbf{Y},k}} \right\rangle}{\mu^{\mathbf{Y},k}\left(t^a \cdot B_0^{\mathbf{Y},k}\right)} < \ell^{kM}$$

The following follows immediately from Corollary 4.2.3(2).

Lemma 6.0.3. *The notion of effectively an-FRS over the origin does not depend on the rectification on the source.*

In the this subsection we will prove the following:

Theorem 6.0.4. *The Chevalley map* p *is effectively an-FRS over the origin.*

Let us briefly explain the idea of the proof of this theorem:

- We first prove 2 statements on the effective an-FRS over the origin property:
  - It is local in the smooth topology (on $\mathbf{X}$). See Corollary 6.1.2
  - For $\mathbb{G}_m$-equivariant maps between affine spaces, one can give a criterion for the effective an-FRS over the origin property in terms of the exponents of the actions of $\mathbb{G}_m$. See Lemma 6.1.1 below.
- We then use the original Harish-Chandra's argument:
  - Proof by a descending induction that the Chevalley map restricted to the complement of an invariant closed subset of $\mathbf{N}$ is effectively an-FRS over the origin.
  - The base of the induction follows from the fact that in this case the origin is not in the image of the map. The fact that such maps are effectively an-FRS over the origin is not completely obvious, but we prove it in Lemma 6.1.3 below.
  - For the step of the induction we use an analog of the Slodowy slice (see Lemma 6.2.1 below). We use the locality of the effective an-FRS over the origin property in order to deduce the statement from an analogous statement for the slice.
  - We prove the analogous statement for the slice using the criterion for the effective an-FRS over the origin property for $\mathbb{G}_m$-equivariant maps.

In the next 2 subsections we provide some preparations for actual proof which will be given in §6.3.

6.1. **Basic properties of effectively an-FRS maps over the origin.**

Lemma 6.1.1. *Let* $\mathbf{X} = \mathbb{A}^I$ *and* $\mathbf{Y} = \mathbb{A}^J$ *be affine spaces with positive actions of* $\mathbb{G}_m$ *given by*

$$s \cdot (x_1, \ldots, x_I) = (s^{a_1} x_1, \ldots, s^{a_I} x_I)$$

and

$$s \cdot (y_1, \ldots, y_J) = (s^{b_1} y_1, \ldots, s^{b_J} y_J)$$

respectively. Assume that $\sum_{i=1}^{I} a_i > \sum_{j=1}^{J} b_j$.

Let $\varphi : \mathbf{X} \to \mathbf{Y}$ *be an equivariant map such that* $\varphi|_{\mathbf{X}\smallsetminus 0} : (\mathbf{X} \smallsetminus 0) \to \mathbf{Y}$ *is effectively an-FRS over the origin. Then* $\varphi : \mathbf{X} \to \mathbf{Y}$ *is effectively an-FRS over the origin.*

Proof. We begin by comparing the balls in $\mathbf{X} \smallsetminus 0$ to spheres in $\mathbf{X}$. More precisely, we fix $m \in \mathbb{N}$. For any $k, i \in \mathbb{N}$, we denote by

$$S_{m,i}^{\mathbf{X},k} := t^i \cdot B_m^{\mathbf{X},k} \smallsetminus t^{i+1} \cdot B_m^{\mathbf{X},k}$$

a sphere in $\mathbf{X}$. To make the comparison, we choose standard rectifications on $\mathbf{X}$ and choose the rectification on $\mathbf{X} \smallsetminus 0$ given by the cover with the complements of coordinate hyperplanes and the induced forms from the standard form on $\mathbf{X}$. It is easy to see that there exists $m' > m$ s.t. for any k we have

$$B_{m'}^{\mathbf{X}\smallsetminus 0,k} \supset S_{m,0}^{\mathbf{X},k}$$

and moreover

$$\ell^{km'} \mu_{m'}^{\mathbf{X}\smallsetminus 0,k} > 1_{S_{m,0}^{\mathbf{X},k}} \mu_m^{\mathbf{X},k}.$$

We choose the standard rectification on $\mathbf{Y}$ and use the above comparison to deduce that for any $k \in \mathbb{N}$ and $a \in \mathbb{Z}$ we have:

$$\frac{\left\langle \varphi_* \left(1_{S_{m,0}^{\mathbf{X},k}} \mu_m^{\mathbf{X},k} \right), 1_{t^a \cdot B_0^{\mathbf{Y},k}} \right\rangle}{\mu^{\mathbf{Y},k}\left(t^a \cdot B_0^{\mathbf{Y},k} \right)} < \ell^{km'} \frac{\left\langle \varphi_* \left(\mu_{m'}^{\mathbf{X}\smallsetminus 0,k} \right), 1_{t^a \cdot B_0^{\mathbf{Y},k}} \right\rangle}{\mu^{\mathbf{Y},k}\left(t^a \cdot B_0^{\mathbf{Y},k} \right)}$$

On the other hand, the assumption that $\varphi|_{\mathbf{X}\smallsetminus 0} : (\mathbf{X} \smallsetminus 0) \to \mathbf{Y}$ is effectively an-FRS over the origin implies that there exists N_0 s.t. for any $k \in \mathbb{N}$ and $a \in \mathbb{Z}$ we have

$$\ell^{km'} \frac{\left\langle \varphi_* \left(\mu_{m'}^{\mathbf{X}\smallsetminus 0,k} \right), 1_{t^a \cdot B_0^{\mathbf{Y},k}} \right\rangle}{\mu^{\mathbf{Y},k}\left(t^a \cdot B_0^{\mathbf{Y},k} \right)} < \ell^{kN_0}$$

Combining the last two inequalities we obtain that for any $k \in \mathbb{N}$ and $a \in \mathbb{Z}$ we have:

$$\frac{\left\langle \varphi_* \left(1_{S_{m,0}^{\mathbf{X},k}} \mu_m^{\mathbf{X},k} \right), 1_{t^a \cdot B_0^{\mathbf{Y},k}} \right\rangle}{\mu^{\mathbf{Y},k}\left(t^a \cdot B_0^{\mathbf{Y},k} \right)} < \ell^{kN_0} \tag{6.1.1}$$

By the equivariance of φ we obtain that for any integers $i, k \in \mathbb{N}$ we have:

$$\varphi_* \left(1_{S_{m,i}^{\mathbf{X},k}} \mu_m^{\mathbf{X},k} \right) = t^i \cdot \left(\varphi_* \left(t^{-i} \cdot \left(1_{S_{m,i}^{\mathbf{X},k}} \mu^{\mathbf{X},k} \right) \right) \right)$$

We also have
$$t^{-i}\cdot\left(1_{S_{m,i}^{\mathbf{X},k}}\mu^{\mathbf{X},k}\right)=\ell^{-ki\sum a_j}1_{S_{m,0}^{\mathbf{X},k}}\mu^{\mathbf{X},k}$$
and similarly
$$t^i\cdot 1_{t^a\cdot B_0^{\mathbf{Y},k}}=1_{t^{a-i}\cdot B_0^{\mathbf{Y},k}} \text{ and } \mu^{\mathbf{Y},k}\left(t^a\cdot B_0^{\mathbf{Y},k}\right)=\ell^{-ki\sum b_j}\mu^{\mathbf{Y},k}\left(t^{a-i}\cdot B_0^{\mathbf{Y},k}\right)$$
Thus, for any integers $i\in\mathbb{N}$, $a\in\mathbb{Z}$, and $k\in\mathbb{N}$, we obtain:
$$\begin{aligned}\frac{\left\langle\varphi_*\left(1_{S_{m,i}^{\mathbf{X},k}}\mu_m^{\mathbf{X},k}\right),1_{t^a\cdot B_0^{\mathbf{Y},k}}\right\rangle}{\mu^{\mathbf{Y},k}\left(t^a\cdot B_0^{\mathbf{Y},k}\right)}&=\ell^{-ki\sum a_j}\frac{\left\langle\varphi_*\left(\left(1_{S_{m,0}^{\mathbf{X},k}}\mu^{\mathbf{X},k}\right)\right),1_{t^{a-i}\cdot B_0^{\mathbf{Y},k}}\right\rangle}{\mu^{\mathbf{Y},k}\left(t^a\cdot B_0^{\mathbf{Y},k}\right)}\\&=\frac{\ell^{-ki\sum a_j}}{\ell^{-ki\sum b_j}}\cdot\frac{\left\langle\varphi_*\left(\left(1_{S_{m,0}^{\mathbf{X},k}}\mu^{\mathbf{X},k}\right)\right),1_{t^{a-i}\cdot B_0^{\mathbf{Y},k}}\right\rangle}{\mu^{\mathbf{Y},k}\left(t^{a-i}\cdot B_0^{\mathbf{Y},k}\right)}\end{aligned}$$
Combining this with the assumption $\sum a_j>\sum b_j$ and the inequality (6.1.1) we obtain
$$\frac{\left\langle\varphi_*\left(1_{S_{m,i}^{\mathbf{X},k}}\mu_m^{\mathbf{X},k}\right),1_{t^a\cdot B_0^{\mathbf{Y},k}}\right\rangle}{\mu^{\mathbf{Y},k}\left(t^a\cdot B_0^{\mathbf{Y},k}\right)}<\ell^{-ki+kN_0}$$

Using the fact that $B_m^{\mathbf{X},k}=\bigsqcup_{i=0}^{\infty}S_{m,i}^{\mathbf{X},k}$ we deduce that for any integers $a\in\mathbb{Z}$, and $k\in\mathbb{N}$, we have:
$$\begin{aligned}\frac{\left\langle\varphi_*\left(\mu_m^{\mathbf{X},k}\right),1_{t^a\cdot B_0^{\mathbf{Y},k}}\right\rangle}{\mu^{\mathbf{Y},k}\left(t^a\cdot B_0^{\mathbf{Y},k}\right)}&=\sum_{i=0}^{\infty}\frac{\left\langle\varphi_*\left(1_{S_{m,i}^{\mathbf{X},k}}\mu_m^{\mathbf{X},k}\right),1_{t^a\cdot B_0^{\mathbf{Y},k}}\right\rangle}{\mu^{\mathbf{Y},k}\left(t^a\cdot B_0^{\mathbf{Y},k}\right)}\\&<\sum_{i=0}^{\infty}\ell^{-ki+kN_0}\leq\ell^{k(N_0+1)}.\end{aligned}$$
Taking $M:=N_0+1$ we get the required bound. ☐

Lemma 4.2.8, Lemma 4.2.10, and Proposition 4.2.7 give us:

Corollary 6.1.2. *Let $\mathbf{X}$ be a μ-rectified variety and let $\gamma:\mathbf{X}\to\mathbf{Y}$ be an algebraic map to an affine space $\mathbf{Y}$. Let $\mathbb{G}_m$ act positively on $\mathbf{Y}$.*

Let $\delta:\tilde{\mathbf{X}}\to\mathbf{X}$ be a submersion.

Then:

1. *(1) if γ is effectively an-FRS over the origin then so is $\gamma\circ\delta$.*
2. *(2) if δ effectively surjective and $\gamma\circ\delta$ is effectively an-FRS over the origin then so is γ.*
3. *(3) if δ is onto on the level of points for any field and $\gamma\circ\delta$ is effectively an-FRS over the origin then γ is also effectively an-FRS over the origin.*

Proof.

(1) Follows from Lemma 4.2.8.
(2) Follows from Lemma 4.2.10.

(3) Follows from (2) and Proposition 4.2.7.

□

Lemma 6.1.3. *Let* $\mathbf{X}$ *be a* μ*-rectified variety and let* $\gamma : \mathbf{X} \to \mathbf{Y}$ *be an algebraic map to an affine space* $\mathbf{Y}$*. Let* $\mathbb{G}_m$ *act positively on* $\mathbf{Y}$*. Assume that* $0 \notin \gamma(\mathbf{X}(\bar{\mathbb{F}}_\ell))$*. Then* γ *is effectively an-FRS over the origin.*

Proof. Fix $m \in \mathbb{N}$. By Corollary 4.2.9, there exists $M_1 \in \mathbb{N}$ s.t. for any $k \in \mathbb{N}$:

$$\mu_m^{\mathbf{X},k}(B_\infty^{\mathbf{X},k}) < \ell^{kM_1}.$$

Let $\lambda_1, \ldots, \lambda_I$ be the exponents of the $\mathbb{G}_m$ action on $\mathbf{Y}$. Choose the rectification on $\mathbf{Y}' := \mathbf{Y} \smallsetminus \{0\}$ given by the open cover of $\mathbf{Y}'$ by compliments to coordinate hyperplanes, and their standard embeddings into $\mathbf{Y}' \times \mathbb{A}^1$. Let $\gamma' : \mathbf{X} \to \mathbf{Y} \smallsetminus \{0\}$ be s.t. γ factors through γ'. By Lemma 4.2.2 there exists m_1 s.t. for any $k \in \mathbb{N}$ we have $\gamma'(B_m^{\mathbf{X},k}) \subset B_{m_1}^{\mathbf{Y}\smallsetminus\{0\},k}$. Take

$$M = M_1 + k(m_1+1)\sum_i \lambda_i + 1.$$

Note that

$$B_{m_1}^{\mathbf{Y}\smallsetminus\{0\},k} = B_{m_1}^{\mathbf{Y},k} \smallsetminus B_{-m_1-1}^{\mathbf{Y},k}(0)$$

Fix $a \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Case 1. $a > \frac{m_1+1}{\min_i \lambda_i}$:

$$\frac{\left\langle \gamma_*\left(\mu_m^{\mathbf{X},k}\right), 1_{t^a \cdot B_0^{\mathbf{Y},k}} \right\rangle}{\mu^{\mathbf{Y},k}\left(t^a \cdot B_0^{\mathbf{Y},k}\right)} \leq \frac{\left\langle \gamma_*\left(\mu_m^{\mathbf{X},k}\right), 1_{B_{-m_1-1}^{\mathbf{Y},k}}(0) \right\rangle}{\mu^{\mathbf{Y},k}\left(t^a \cdot B_0^{\mathbf{Y},k}\right)} = 0 < \ell^{kM}$$

Case 2. $a \leq \frac{m_1+1}{\min_i \lambda_i}$:

$$\frac{\left\langle \gamma_*\left(\mu_m^{\mathbf{X},k}\right), 1_{t^a \cdot B_0^{\mathbf{Y},k}} \right\rangle}{\mu^{\mathbf{Y},k}\left(t^a \cdot B_0^{\mathbf{Y},k}\right)} \leq \frac{\mu_m^{\mathbf{X},k}(\mathbf{X}(\mathbb{F}_{\ell^k}((t))))}{\mu^{\mathbf{Y},k}\left(t^{m_1+1} \cdot B_0^{\mathbf{Y},k}\right)} \leq \frac{\ell^{kM_1}}{\ell^{-k(m_1+1)\sum_i \lambda_i}} < \ell^{kM}$$

□

6.2. **Slices to nilpotent elements.** We will need existence of slices for nilpotent orbits in $\mathfrak{g}$ that have a $\mathbb{G}_m$ action satisfying an appropriate condition. This is analogous to the approach of Harish-Chandra's in his proof of the boundedness of normalized orbital integrals. There the slice was defined by an $\mathfrak{sl}_2$ triple (a.k.a. Slodowy slice). However, we can not rely on $\mathfrak{sl}_2$ triples, so we have to do it in a more ad-hoc procedure.

Lemma 6.2.1. *Let* $x \in \underline{\mathfrak{g}}(\mathbb{F}_\ell)$ *be a nilpotent element. Then there are:*

- *a linear subspace* $\mathbf{L} \subset \underline{\mathfrak{g}}$
- *a positive* $\mathbb{G}_m$ *action on* $\mathbf{L}$*.*

s.t.

(1) the action map $\mathbf{G} \times (x + \mathbf{L}) \to \underline{\mathfrak{g}}$ *is a submersion.*
(2) For any nilpotent $y \in x + (\mathbf{L}(\bar{\mathbb{F}}_\ell) \smallsetminus 0)$ *we have* $\dim \mathbf{G} \cdot y > \dim \mathbf{G} \cdot x$

(3) The Chevalley map $p|_{x+\mathbf{L}}$ intertwines the $\mathbb{G}_m$ action on $x+\mathbf{L}$ (given by the identification $y \mapsto x+y$ between $\mathbf{L}$ and $x+\mathbf{L}$) with the $\mathbb{G}_m$ action on $\underline{\mathfrak{c}}$ given by $(\lambda \cdot f)(z) := \lambda^n f(\lambda^{-1} \cdot z)$.
(4) If x is not regular then the sum of the exponents of the $\mathbb{G}_m$ action on $\mathbf{L}$ is larger than $\frac{n(n+1)}{2}$.

Remark 6.2.2. *Similar results were proven in various contexts, see e.g.* [Pre03]*. Since we would like to have an explicit construction, we include the proof here for completeness.*

For the proof we will need some notations:

Notation 6.2.3.

(1) Consider the action of $\mathbf{G}(F) \times F^\times$ on $\underline{\mathfrak{g}}(F)$ where the first coordinate acts by conjugation and the second by homothety. We denote this action by $(g, \lambda) \cdot A = \lambda^{-1} g A g^{-1}$.
(2) For an integer $k > 0$ let J_k be the $k \times k$ nilpotent Jordan block.
(3) Given a matrix of of matrices $\{A_{i,j}\}_{i,j \in \{1,\dots,j\}}$ we denote by $Bl(\{A_{i,j}\})$ the corresponding block matrix. Here we assume that the sizes of $A_{i,j}$ matches.
(4) For 2 integers n_1, n_2 set

$$\mathbf{V}'_{n_1,n_2} := \{\{a_{kl}\} \in Mat_{n_1,n_2} | a_{kl} = 0 \text{ for } l > 1\}.$$

(5) For an integer k we set $t_k : \mathbb{G}_m \to \mathrm{GL}_k$ defined by

$$t_k(\lambda) = \mathrm{diag}(1, \lambda, \dots, \lambda^{k-1}).$$

(6) For a nilpotent element in Jordan canonical form $x \in \underline{\mathfrak{g}}(\mathbb{F}_\ell)$ with blocks of sizes $n_1, \dots, n_k$ we define maps $t_{i,j} : \mathbb{G}_m \to \mathrm{Mat}_{n_i \times n_j}$ by

$$t_{i,j}(\lambda) := \begin{cases} t_{n_i}(\lambda) & \text{for } i = j \\ 0 & \text{otherwise} \end{cases}$$

Define $t_x : \mathbb{G}_m \to \mathbf{G}$ by

$$t_x(\lambda) = Bl(\{t_{i,j}(\lambda)\}).$$

(7) For x as above define $\phi_x : \mathbb{G}_m \to \mathbf{G} \times \mathbb{G}_m$ by

$$\phi_x(\lambda) = (t_x(\lambda), \lambda^{-1}).$$

(8) For x as above, define

$$\mathbf{L}_x = \{Bl(\{A_{i,j}\}_{i,j}) | A_{i,j} \in \mathbf{V}_{n_i,n_j}\} \subset \underline{\mathfrak{g}}.$$

Proof of Lemma 6.2.1. We now construct our slices. WLOG we can assume that x is a Jordan canonical form. We define:

(i) A linear space $\mathbf{L} := \mathbf{L}_x$, see Notation 6.2.3(8).

(ii) An action of $\mathbb{G}_m$ on $\mathbf{L}$ by

$$\lambda \star A := \phi_x(\lambda) \cdot A,$$

see Notation 6.2.3(1,7).

It is easy to see that this is a positive action. It is evident that condition (3) is satisfied:

$$p(\lambda \star A) = p(\phi_x(\lambda) \cdot A)) = p(\lambda Ad(t_x(\lambda))(A)) = p(Ad(t_x(\lambda))(\lambda A)) = p(\lambda A) = \lambda \cdot p(A).$$

We now verify condition (4). Assume x is not regular, and let $n_1 \geq n_2 \cdots \geq n_k$ be the corresponding partition of n, with $k \geq 2$.

It is a simple verification that:

(a) In the block $n_i \times n_i$ the exponents are $1, 2, \cdots, n_i$.

(b) For $i < j$ we have two blocks $n_i \times n_j$ where the exponents are

$$n_i - n_j + 1, \cdots, n_i.$$

(c) For $i > j$ we have two blocks $n_i \times n_j$ where the exponents are

$$1, \cdots, n_j.$$

To sum up we obtain that the sum of the $\mathbb{G}_m$ exponents is:

$$\sum_{i=1}^{k} \frac{1}{2} n_i(n_i+1) + \sum_{1 \leq i < j \leq k} (n_i n_j + n_j) = \frac{1}{2}\left(\sum_{i=1}^{k} n_i(n_i+1) + \sum_{1 \leq i < j \leq k} (2n_i n_j + 2n_j)\right)$$

$$= \frac{1}{2}(n^2 + n) + \sum_{j=1}^{k} (j-1) n_j > \frac{1}{2}(n^2 + n).$$

It remains to prove conditions (1,2). For (1) we first note that the action map is submersive at point $(1, x)$. Then we deduce (1) from the $\mathbb{G}_m$ action $\star$.

For (2) take a nilpotent element $y \in x + (\mathbf{L}(\bar{\mathbb{F}}_\ell) \smallsetminus 0)$. Using the action $\star$ we see that $\overline{(\mathbf{G} \times \mathbb{G}_m) \cdot y} \ni x$. Since y is nilpotent, we have $(\mathbf{G} \times \mathbb{G}_m) \cdot y = \mathbf{G} \cdot y$. Thus $\overline{\mathbf{G} \cdot y} \ni x$. Assume, for the contradiction, that $\dim(\mathbf{G} \cdot y) \leq \dim(\mathbf{G} \cdot x)$. We obtain $\mathbf{G} \cdot y \ni x$. It remains to show that $\mathbf{G} \cdot x \cap x + \mathbf{L}_x = \{x\}$. This follows from the following easily verified facts:

- x is an isolated point of the intersection $\mathbf{G} \cdot x \cap x + \mathbf{L}_x$.
- the intersection $\mathbf{G} \cdot x \cap x + \mathbf{L}_x$ is $\mathbb{G}_m$-invariant w.r.t. the action $\star$ and the closure of any such G_m-orbit of a point in the slice $x + \mathbf{L}_x$ includes x.

To verify the first point we claim that the intersection of the tangent space $T_x(G \cdot x)$ and L_x is zero. For $x = J_n$ this follows from the following statement: If the matrix $[J_n, A]$ that has the first column equals to zero then $[J_n, A] = 0$. A similar argument works for any direct sum of Jordan blocks. $\square$

6.3. **Proof of Theorem 6.0.4.**

Proof. Let $\mathbf{N} \subset \underline{\mathfrak{g}}$ be the nilpotent cone. Enumerate the nilpotent orbits $\{0\} = \mathbf{O}_1, \dots \mathbf{O}_m$ s.t. $\dim \mathbf{O}_i \le \dim \mathbf{O}_j$ for any $i < j$. Let $\mathbf{N}_i = \bigcup_{j=1}^{i} \mathbf{O}_j$. Note that $\mathbf{N}_i$ is closed and $\mathbf{N}_0 = \emptyset$. We will prove by down going induction on i that for any $i \ge 0$ the map $p|_{\underline{\mathfrak{g}} \smallsetminus \mathbf{N}_i}$ is effectively an-FRS over the origin.

The base of the induction $i = m$ follows from Lemma 6.1.3.

For the induction step, we assume the statement holds for $\mathbf{N}_{i+1}$ and prove it for $\mathbf{N}_i$. Let $\mathbf{U} = \underline{\mathfrak{g}} \smallsetminus \mathbf{N}_{i+1}$. Let $x \in \mathbf{O}_{i+1}(\mathbb{F}_\ell)$ and let $\mathbf{L}$ be the linear space given by Lemma 6.2.1 when applied to x.

Step 1. Reduction to $\mathbf{G} \times \mathbf{L}$.

Consider the map

$$\delta : (\mathbf{G} \times \mathbf{L}) \sqcup \mathbf{U} \to \underline{\mathfrak{g}} \smallsetminus \mathbf{N}_i$$

given on $\mathbf{G} \times \mathbf{L}$ by $\delta(g, l) := g \cdot (x + l)$ and on $\mathbf{U}$ by the embedding $\mathbf{U} \subset \underline{\mathfrak{g}} \smallsetminus \mathbf{N}_i$. By Lemma 6.2.1 (1) δ is submersive. Also, it is onto on the level of points over any field. Indeed, for any extension $E/\mathbb{F}_\ell$ we have $\mathbf{O}_{i+1}(E) = \mathbf{G}(E) \cdot x$ and thus,

$$(\underline{\mathfrak{g}} \smallsetminus \mathbf{N}_i)(E) = \mathbf{O}_{i+1}(E) \cup \mathbf{U}(E) = (\mathbf{G}(E) \cdot x) \cup \mathbf{U}(E) \subset \delta(((\mathbf{G} \times \mathbf{L}) \sqcup \mathbf{U})(E))$$

Thus by Corollary 6.1.2 it is enough to show that $p \circ \delta$ is effectively an-FRS over the origin. Let $\delta' := \delta|_{\mathbf{G} \times \mathbf{L}}$. Notice that $p|_{\mathbf{U}}$ is effectively an-FRS over the origin by the induction hypothesis.

Therefore it is enough to show that $p \circ \delta'$ is effectively an-FRS over the origin.

Notice that $\delta'(\mathbf{G} \times (\mathbf{L} \smallsetminus 0)) \subset \mathbf{U}$ by Lemma 6.2.1 (2). So, by Corollary 6.1.2 we deduce that $p \circ \delta'|_{\mathbf{G} \times (\mathbf{L} \smallsetminus 0)}$ is effectively an-FRS over the origin.

Step 2. Reduction to $\mathbf{L}$.

We can factor the map $p \circ \delta'$ as $p|_{(x+\mathbf{L})} \circ sh_x \circ pr_{\mathbf{L}}$, where $sh_x : \mathbf{L} \to x + \mathbf{L}$ is the shift map, and $pr_{\mathbf{L}} : \mathbf{G} \times \mathbf{L} \to \mathbf{L}$ is the projection. So, by Corollary 6.1.2 it is enough to show that $p|_{x+\mathbf{L}} \circ sh_x : \mathbf{L} \to \mathfrak{c}$ is effectively an-FRS over the origin.

Also, by Corollary 6.1.2 we deduce that $p|_{x+(\mathbf{L} \smallsetminus 0)} \circ sh_x$ is effectively an-FRS over the origin.

Step 3. Proof that $p|_{x+\mathbf{L}} \circ sh_x : \mathbf{L} \to \mathfrak{c}$ is effectively an-FRS over the origin.

If x is regular nilpotent then $p|_{x+\mathbf{L}} \circ sh_x$ is an isomorphism, and thus is effectively an-FRS over the origin.

Otherwise, the assertion follows now from Lemma 6.1.1, since the condition of Lemma 6.1.1 on the source $\mathbf{L}$ is given by Lemma 6.2.1(4).

□

7. An-FRS maps and jets

In this section we relate the effective an-FRS property to jet schemes. Specifically we prove:

Proposition 7.0.1. *Let $\gamma : \mathbf{X} \to \mathbf{Y}$ be a flat $\mathbb{G}_m$-equivariant map between affine spaces with positive $\mathbb{G}_m$ actions. Assume that it is effectively an-FRS over the origin. Then there exists $C \in \mathbb{N}$ s.t. for every $m \in \mathbb{N}$ we have*

$$\dim \mathcal{J}_m(\gamma^{-1}(0)) < C + m \dim \gamma^{-1}(0).$$

Idea of the proof. The fact that γ is effectively an-FRS over the origin provides a bound on the ratio between the measure of the preimage of a certain ellipsoid in $\mathbf{Y}$ and the measure of the ellipsoid itself. If this ellipsoid would be a ball then this ratio would be exactly the (normalized) number of points of the jet scheme in question. So, we would get the desired bound on the dimension from the Lang-Weil bounds. In our case, we get a bound on the dimension of some other scheme ($\delta^{-1}(0)$ in the notation below). We can embed our ellipsoid into a ball. This ball is the union of shifts of the ellipsoid. So we need to bound the measures of the preimages of these shifts. The Lang-Weil bounds translate the problem to a question on dimensions of neighboring fibers of the same map δ. We can bound these dimensions using the semi-continuity of the dimension of the fiber. □

Proof. Choose the standard rectifications on $\mathbf{X}$ and $\mathbf{Y}$. Since γ is effectively an-FRS over the origin, there exists $M \in \mathbb{N}$ such that for any $k, m \in \mathbb{N}$ we have

$$\frac{\left\langle \gamma_* \left(\mu_0^{\mathbf{X},k}\right), 1_{t^m \cdot B_0^{\mathbf{Y},k}} \right\rangle}{\mu^{\mathbf{Y},k}\left(t^m \cdot B_0^{\mathbf{Y},k}\right)} < \ell^{kM}.$$

Let $n := \dim \mathbf{Y}$ and let $\{a_i\}_{i=1}^n$ be the exponents of the $\mathbb{G}_m$ action on $\mathbf{Y}$. Set $a_{\max} = \max a_i$ and $a_{\min} = \min a_i$. Take $C := M + a_{\min} \dim(\mathbf{X})$.

For $m \in \mathbb{N}$ consider the jet map $\mathcal{J}_m(\gamma) : \mathcal{J}_m(\mathbf{X}) \to \mathcal{J}_m(\mathbf{Y})$. Let

$$E_m^k := \left\{ (\sum_j y_{1j} t^j, \ldots, \sum_j y_{nj} t^j) \in \mathcal{J}_{a_{\max} m}(\mathbf{Y})(\mathbb{F}_{\ell^k}) \mid y_{ij} = 0 \text{ for } j < m a_i \right\}.$$

We observe that the measure of an ellipsoid in $\mathbf{Y}$ can be calculated by counting points of this set. Namely,

$$\mu^{\mathbf{Y},k}\left(t^m \cdot B_0^{\mathbf{Y},k}\right) = \frac{\# E_m^k}{\ell^{k m a_{\max} \dim(\mathbf{Y})}}.$$

Also observe that

$$\left\langle \gamma_* \left(\mu_0^{\mathbf{X},k}\right), 1_{t^m \cdot B_0^{\mathbf{Y},k}} \right\rangle = \mu_0^{\mathbf{X},k}(\gamma^{-1}(t^m \cdot B_0^{\mathbf{Y},k})) = \frac{\# \mathcal{J}_{m a_{\max}}(\gamma)^{-1}(E_m^k)}{\ell^{k m a_{\max} \dim(\mathbf{X})}}.$$

We get

$$\begin{aligned}
\#\mathcal{J}_{ma_{\max}}(\gamma)^{-1}(E_m^k) &= \ell^{kma_{\max}\dim(\mathbf{X})}\left\langle \gamma_*\left(\mu_0^{\mathbf{X},k}\right), 1_{t^m\cdot B_0^{\mathbf{Y},k}}\right\rangle \\
&= \ell^{kma_{\max}\dim(\mathbf{X})}\frac{\left\langle \gamma_*\left(\mu_0^{\mathbf{X},k}\right), 1_{t^m\cdot B_0^{\mathbf{Y},k}}\right\rangle}{\mu^{\mathbf{Y},k}\left(t^m\cdot B_0^{\mathbf{Y},k}\right)}\mu^{\mathbf{Y},k}\left(t^m\cdot B_0^{\mathbf{Y},k}\right) \\
&= \ell^{kma_{\max}(\dim(\mathbf{X})-\dim(\mathbf{Y}))}\frac{\left\langle \gamma_*\left(\mu_0^{\mathbf{X},k}\right), 1_{t^m\cdot B_0^{\mathbf{Y},k}}\right\rangle}{\mu^{\mathbf{Y},k}\left(t^m\cdot B_0^{\mathbf{Y},k}\right)}\#E_m^k \\
&< \ell^{kma_{\max}\dim(\gamma^{-1}(0))}\ell^{kM}\#E_m^k,
\end{aligned}$$

where in the last step we used the fact that γ is flat, as well as the inequality established above.

Let $\mathbf{E}_m \subset \mathcal{J}_{ma_{\max}}(\mathbf{Y})$ be the natural subvariety s.t. $\mathbf{E}_m(\mathbb{F}_{\ell^k}) = E_m^k$. More precisely,

$$\mathbf{E}_m := \left\{(\sum_j y_{1j}t^j,\dots,\sum_j y_{nj}t^j) \in \mathcal{J}_{a_{\max}m}(\mathbf{Y}) \mid y_{ij}=0 \text{ for } 1\le i\le n \text{ and } j<ma_i\right\}.$$

By the Lang-Weil bounds [LW54, Theorem 1], we obtain

$$\begin{aligned}
\dim\mathcal{J}_{ma_{\max}}(\gamma)^{-1}(\mathbf{E}_m) &= \limsup_{k\to\infty}\log_{\ell^k}(\#\mathcal{J}_{ma_{\max}}(\gamma)^{-1}(\mathbf{E}_m))(\mathbb{F}_{\ell^k}) \le \\
&\le M + ma_{\max}\dim(\gamma^{-1}(0)) + \limsup_{k\to\infty}\log_{\ell^k}(\#\mathbf{E}_m(\mathbb{F}_{\ell^k})) = \\
&= M + ma_{\max}\dim(\gamma^{-1}(0)) + \dim(\mathbf{E}_m)
\end{aligned}$$

Note that $\mathbf{E}_m \subset \mathcal{J}_{ma_{\max}}(\mathbf{Y})$ has an algebraic group structure coming from the additive structure on $\mathbf{Y}$.

The action of $\mathbb{G}_m$ on $\mathbf{Y}$ induce an action on $\mathbf{E}_m$ and $\mathcal{J}_{ma_{\max}}(\mathbf{Y})$. Let $\mathbf{Z} := \mathcal{J}_{ma_{\max}}(\mathbf{Y})/\mathbf{E}_m$ and let $\varepsilon : \mathcal{J}_{ma_{\max}}(\mathbf{Y}) \to \mathbf{Z}$ be the quotient map. Consider the map

$$\delta := \varepsilon\circ\mathcal{J}_{ma_{\max}}(\gamma) : \mathcal{J}_{ma_{\max}}(\mathbf{X}) \to \mathbf{Z}.$$

We have

$$\dim(\delta^{-1}(0)) = \dim\mathcal{J}_{ma_{\max}}(\gamma)^{-1}(\mathbf{E}_m) \le M + ma_{\max}\dim(\gamma^{-1}(0)) + \dim(\mathbf{E}_m)$$

By upper semi-continuity of the dimension of the fiber ([DG67, IV 13.1.3,13.1.15]), we have a Zariski open $\mathbf{U}\subset\mathbf{Z}$ s.t. for any $x\in\mathbf{U}(\bar{\mathbb{F}}_\ell)$ we have:

$$\dim(\delta^{-1}(x)) \le \dim(\delta^{-1}(0)) \le M + ma_{\max}\dim(\gamma^{-1}(0)) + \dim(\mathbf{E}_m) \tag{7.0.1}$$

Using the action of $\mathbb{G}_m$ on $\mathbf{Z}$ we obtain that (7.0.1) holds for any $x\in\mathbf{Z}(\bar{\mathbb{F}}_\ell)$.

Let

$$\mathbf{R}_m := \{(\sum_j y_{1j}t^j,\dots,\sum_j y_{nj}t^j) \in \mathcal{J}_{a_{\max}m}(\mathbf{Y}) | y_{ij}=0 \text{ for } j<ma_{\min}\}.$$

Set $\mathbf{W} = \mathbf{R}_m/\mathbf{E}_m$ with the natural embedding to $\mathbf{Z}$. Using (7.0.1) we deduce

$$\begin{aligned}\dim \mathcal{J}_{ma_{\max}}(\gamma)^{-1}(\mathbf{R}_m) = \dim(\delta^{-1}(\mathbf{W})) \leq \\ &\leq M + ma_{\max}\dim(\gamma^{-1}(0)) + \dim(\mathbf{E}_m) + \dim(\mathbf{W}) = \\ &= M + ma_{\max}\dim(\gamma^{-1}(0)) + \dim(\mathbf{R}_m)\end{aligned}$$

Consider the commutative diagram.

$$\begin{array}{ccc} \mathcal{J}_{ma_{\max}}(\mathbf{X}) & \xrightarrow{\mathcal{J}_{ma_{\max}}(\gamma)} & \mathcal{J}_{ma_{\max}}(\mathbf{Y}) \\ {\scriptstyle r_{\mathbf{X}}}\downarrow & & \downarrow{\scriptstyle r_{\mathbf{Y}}} \\ \mathcal{J}_{ma_{\min}}(\mathbf{X}) & \xrightarrow{\mathcal{J}_{ma_{\min}}(\gamma)} & \mathcal{J}_{ma_{\min}}(\mathbf{Y}) \end{array}$$

were the vertical maps are the reduction maps. We note that $\mathbf{R}_m = r_{\mathbf{Y}}^{-1}(0)$. Clearly,

$$\dim(r_{\mathbf{Y}}^{-1}(0)) - \dim(r_{\mathbf{X}}^{-1}(0)) = m(a_{\min} - a_{\max})\dim(\gamma^{-1}(0))$$

We have

$$\mathcal{J}_{ma_{\max}}(\gamma)^{-1}(\mathbf{R}_m) = (r_{\mathbf{Y}} \circ \mathcal{J}_{ma_{\max}}(\gamma))^{-1}(0) = (\mathcal{J}_{ma_{\min}}(\gamma) \circ r_{\mathbf{X}})^{-1}(0) = r_{\mathbf{X}}^{-1}((\mathcal{J}_{ma_{\min}}(\gamma))^{-1}(0)).$$

Since $r_{\mathbf{X}}$ is a quotient map, we get

$$\begin{aligned}\dim(\mathcal{J}_{ma_{\min}}(\gamma)^{-1}(0)) &= \dim(r_X^{-1}((\mathcal{J}_{ma_{\min}}(\gamma))^{-1}(0))) - \dim(r_X^{-1}(0)) \\ &= \dim(\mathcal{J}_{ma_{\max}}(\gamma)^{-1}(\mathbf{R}_m)) - \dim(r_{\mathbf{X}}^{-1}(0)) \\ &\leq M + ma_{\max}\dim(\gamma^{-1}(0)) + \dim(\mathbf{R}_m) - \dim(r_{\mathbf{X}}^{-1}(0)) \\ &= M + ma_{\max}\dim(\gamma^{-1}(0)) + \dim(r_{\mathbf{Y}}^{-1}(0)) - \dim(r_{\mathbf{X}}^{-1}(0)) \\ &= M + ma_{\min}\dim(\gamma^{-1}(0))\end{aligned}$$

Now, fix m'. Let m be the largest integer s.t. $ma_{\min} < m'$, that is $m = [\frac{m'}{a_{\min}}]$. Consider the restriction map $\mathcal{J}_{m'}(\gamma^{-1}(0)) \to \mathcal{J}_{ma_{\min}}(\gamma^{-1}(0))$. Its fibers are of dimension $\leq (m' - ma_{\min})\dim \mathbf{X} \leq a_{\min}\dim \mathbf{X}$. We obtain

$$\begin{aligned}\dim \mathcal{J}_{m'}(\gamma^{-1}(0)) \leq a_{\min}\dim \mathbf{X} + \dim \mathcal{J}_{ma_{\min}}(\gamma^{-1}(0)) = a_{\min}\dim \mathbf{X} + \dim \mathcal{J}_{ma_{\min}}(\gamma)^{-1}(0) \leq \\ &\leq a_{\min}\dim \mathbf{X} + M + ma_{\min}\dim(\gamma^{-1}(0)) \leq \\ &\leq a_{\min}\dim \mathbf{X} + M + m'\dim(\gamma^{-1}(0)) = C + m'\dim(\gamma^{-1}(0)).\end{aligned}$$

□

8. Proof of Theorems A, B, and C

Proof of Theorem A. According to Theorem 6.0.4 the Chevalley map $p : \mathfrak{g} \to \mathfrak{c}$ is effectively an-FRS over the origin. We apply Proposition 7.0.1 with p and as $p^{-1}(0) = \mathbf{N}$, we obtain the result. □

Proof of Theorem B . Let C_0 be as in Theorem A. Fix k. Take $C = C_0$. By semi-continuity of the dimension of the fiber ([DG67, IV 13.1.3,13.1.15]), we have a Zariski open $\mathbf{U} \subset \mathcal{J}_m(\mathfrak{c})$ s.t. for any $x \in \mathbf{U}(\bar{\mathbb{F}}_\ell)$ we have:

(8.0.1) $$\dim(\mathcal{J}_m(p)^{-1}(x)) \leq \dim(p^{-1}(0)) \leq C + m\dim(p^{-1}(0)).$$

Using the action of $\mathbb{G}_m$ on $\mathcal{J}_m(\underline{\mathfrak{c}})$ and $\mathcal{J}_m(\underline{\mathfrak{g}})$ we obtain that (8.0.1) holds for any $x \in \mathcal{J}_m(\underline{\mathfrak{c}})(\bar{\mathbb{F}}_\ell)$ as required. □

Proof of Theorem C. Let C be as in Theorem B. Fix i and let $C_i := iC$. Let $p_i : \underline{\mathfrak{g}}_i \to \underline{\mathfrak{c}}$ be the projection. We deduce that for any $x \in \mathcal{J}_m(\underline{\mathfrak{c}})(\bar{\mathbb{F}}_\ell)$ we have:

$$\begin{aligned}\dim(\mathcal{J}_m(p_i)^{-1}(x)) = i\dim(\mathcal{J}_m(p)^{-1}(x)) \leq \\ \leq iC + mi\dim(p^{-1}(0)) = \\ = iC + m\dim(p_i^{-1}(0))\end{aligned}$$

Note that by Corollary 5.0.5 the map p_i is flat. Thus

$$\begin{aligned}\dim(\mathcal{J}_m(\underline{\mathfrak{g}}^{\times_{\underline{\mathfrak{c}}} i})) = \dim(\mathcal{J}_m(p_i)^{-1}(x)) + \dim(\mathcal{J}_m(\underline{\mathfrak{c}})) \leq \\ \leq iC + m\dim(p_i^{-1}(0)) + \dim(\mathcal{J}_m(\underline{\mathfrak{c}})) = C_i + m\dim(\underline{\mathfrak{g}}^{\times_{\underline{\mathfrak{c}}} k}).\end{aligned}$$

□

9. Almost integrability

In this section we study several versions of integrability of an algebraic variety. We will prove that they are equivalent under the assumption of existence of a resolution (see Theorem 9.0.3).

Definition 9.0.1. *Let $\mathbf{X}$ be a variety and $\mathbf{U} \subset \mathbf{X}^{sm}$ be an open subset. We say that $(\mathbf{X}, \mathbf{U})$ is:*

(1) *asymptotically almost integrable if for any:*
   - *open affine $\mathbf{V} \subset \mathbf{X}$*
   - *top form ω on $\mathbf{V}^{sm}$*

   there is $M \in \mathbb{N}$ s.t. for any $m' \in \mathbb{N}$ and any rectification of $\mathbf{U} \cap \mathbf{V}$ we have
$$\lim_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_{m'}^{\mathbf{U}\cap\mathbf{V},k} \cap \mathbf{V}(\mathbb{F}_{\ell^k}[[t]])} |\omega| = 0.$$
(2) *Geometrically almost integrable if for any open affine (rectified) $\mathbf{V} \subset \mathbf{X}$ and any top form ω on $\mathbf{V}^{sm}$ there is a resolution of singularities $\tilde{\mathbf{V}} \to \mathbf{V}$, which is an isomorphism over $\mathbf{V}^{sm}$, s.t. that one can extend ω to a rational form on $\tilde{\mathbf{V}}$ whose poles form an SNC divisor with multiplicities 1.*
(3) *Analytically almost integrable if for any*
   - *open affine $\mathbf{V} \subset \mathbf{X}$,*
   - *a rectification on $\mathbf{V} \cap \mathbf{U}$,*
   - *a top form ω on $\mathbf{V}^{sm}$,*
   - *$k \in \mathbb{N}$,*

   there is $M \in \mathbb{N}$ s.t. for any $m \in \mathbb{N}$ we have
$$\int_{B_m^{\mathbf{U}\cap\mathbf{V},k} \cap \mathbf{V}(\mathbb{F}_{\ell^k}[[t]])} |\omega| < M(m+1)^M.$$

Remark 9.0.2. *The notion of geometrically almost integrable does not depend on U.*

Theorem 9.0.3. *Assume that* $\mathbf{X}$ *has a strong resolution (See §2.1(11)). Then TFAE:*

(1) $(\mathbf{X}, \mathbf{U})$ *is asymptotically almost integrable.*
(2) $(\mathbf{X}, \mathbf{U})$ *is geometrically almost integrable.*
(3) $(\mathbf{X}, \mathbf{U})$ *is analytically almost integrable.*

Idea of the proof. We assume WLOG $\mathbf{V} = \mathbf{X}$.

(1) or (3) $\Rightarrow$ (2): We are given a strong resolution of $\mathbf{X}$. We have to show that the poles of ω after the pull-back to this resolution are simple. We assume the contrary, and take a generic point x on the divisor where ω have a non-simple pole. We replace the integral in (1) or (3) by an integral over a small ball around x. This is a smaller integral so the bound in (1) or (3) still valid for it. On the other hand we can compute it using local coordinates near this point and using the knowledge on the pole of ω. A simple computation contradicts the bound provided by (1) or (3).

(2) $\Rightarrow$ (1) and (3): We are given a resolution where all the poles of ω are simple. The integral in (1) and (3) can be computed on this resolution. This can be done locally. The local computation is a computation of an integral of a monomial top form on $\mathbb{A}^n$ over a ball in the complement to a union of coordinate hyperplanes. This computation gives the required bound.

□

Proof.

(1)$\Rightarrow$(2): Let $\gamma : \tilde{\mathbf{X}} \to \mathbf{X}$ be a strong resolution. Let $\mathbf{V}, \omega$ be as in Definition 9.0.1(2). WLOG assume $\mathbf{X} = \mathbf{V}$. Let $\mathbf{Z} = \gamma^{-1}(\mathbf{X} \smallsetminus \mathbf{U})$ and $\tilde{\mathbf{U}} := \gamma^{-1}(\mathbf{U})$. Note that $\tilde{\mathbf{U}} \cong \mathbf{U}$. Take $\tilde{\mathbf{V}} = \tilde{\mathbf{X}}$.

Let $\mathbf{Z}' \subset \mathbf{Z}$ be the exceptional divisor. We have $\mathbf{Z} = \mathbf{Z}' \cup \mathbf{Z}''$ where $\mathbf{Z}', \mathbf{Z}'' \subset \mathbf{Z}$ are closed and have no common components. Let $\tilde{\omega} := \gamma^*(\omega)$ considered as a rational form on $\tilde{\mathbf{X}}$.

The support of the poles of $\tilde{\omega}$ is contained in $\mathbf{Z}'$ which is an SNC divisor. Assume for the sake of contradiction that not all the poles are simple. Let $\mathbf{Z}_0$ be a component of $\mathbf{Z}'$ where $\tilde{\omega}$ have a pole of multiplicity $f > 1$. Let $Z''' \subset \tilde{\mathbf{X}}$ be (the closure of) the zero-locus of $\tilde{\omega}$.

Replacing ℓ with its power if necessary we may assume that there is $z \in \mathbf{Z}_0(\mathbb{F}_\ell)$ that is outside all the other components of $\mathbf{Z}$ and $\mathbf{Z}'''$.

Let $\mathbf{V}'$ be an affine open neighborhood of z that does not intersect the other components of $\mathbf{Z}$ and $\mathbf{Z}'''$. Choose a rectification of $\mathbf{U}$. Using the identification $\tilde{\mathbf{U}} \cong \mathbf{U}$ we obtain a rectification of $\tilde{\mathbf{U}}$. Choose a rectification of $\mathbf{V}' \smallsetminus \mathbf{Z}_0 = \tilde{\mathbf{U}} \cap \mathbf{V}'$ s.t. for any $m, k \in \mathbb{N}$ we have

$$B_m^{\tilde{\mathbf{U}} \cap \mathbf{V}', k} \subset B_m^{\tilde{\mathbf{U}}, k}.$$

Finally, choose an arbitrary rectification on $\mathbf{V}'$.

Since γ is defined over $\mathbb{F}_\ell$, for any k we have

$$\gamma(\tilde{\mathbf{X}}(\mathbb{F}_{\ell^k}[[t]])) \subset \mathbf{X}(\mathbb{F}_{\ell^k}[[t]]).$$

By the assumption, there is $M \in \mathbb{N}$ s.t. for any $m' \in \mathbb{N}$ we have

$$\lim_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_{m'}^{\mathbf{U},k} \cap \mathbf{X}(\mathbb{F}_{\ell^k}[[t]])} |\omega| = 0.$$

For any $m' \in \mathbb{N}$, we obtain:

$$0 \leq \limsup_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_{m'}^{\mathbf{V}'\smallsetminus\mathbf{Z}_0,k} \cap B_{-1}^{\mathbf{V}',k}(z)} |\tilde{\omega}|$$
$$\leq \limsup_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_{m'}^{\mathbf{V}'\smallsetminus\mathbf{Z}_0,k} \cap \tilde{\mathbf{X}}(\mathbb{F}_{\ell^k}[[t]])} |\tilde{\omega}| = \limsup_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_{m'}^{\mathbf{V}'\cap\tilde{\mathbf{U}},k} \cap \tilde{\mathbf{X}}(\mathbb{F}_{\ell^k}[[t]])} |\tilde{\omega}|$$
$$\leq \limsup_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_{m'}^{\tilde{\mathbf{U}},k} \cap \tilde{\mathbf{X}}(\mathbb{F}_{\ell^k}[[t]])} |\tilde{\omega}| \leq \limsup_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_{m'}^{\mathbf{U},k} \cap \mathbf{X}(\mathbb{F}_{\ell^k}[[t]])} |\omega| = 0.$$

Thus for any $m' \in \mathbb{N}$ we have:

$$\lim_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_{m'}^{\mathbf{V}'\smallsetminus\mathbf{Z}_0,k} \cap B_{-1}^{\mathbf{V}',k}(z)} |\tilde{\omega}| = 0.$$

Note that this statement is independent of the rectification on $\mathbf{V}'$.

Since $\mathbf{Z}_0$ is smooth, we can choose an affine $\mathbf{V}'' \subset \mathbf{V}'$ and an etale map $\phi : \mathbf{V}'' \to \mathbb{A}^{\dim \mathbf{X}}$ s.t.

– $z \in \mathbf{V}''(\mathbb{F}_\ell)$

– $\phi(z) = 0$

– $\phi^{-1}(\mathbb{A}^{\dim \mathbf{X}-1}) = \mathbf{Z}_0 \cap \mathbf{V}''$

Let x_1 be the defining coordinate of $\mathbb{A}^{\dim \mathbf{X}-1} \subset \mathbb{A}^{\dim \mathbf{X}}$. Note that

$$\phi^* \left(\frac{dx_1 \wedge \cdots \wedge dx_2}{x_1^f} \right) = g\tilde{\omega},$$

where $g \in O_{\mathbf{V}''}^\times(\mathbf{V}'')$.

Choose an arbitrary simple rectification on $\mathbf{V}''$ and choose the rectification on $\mathbf{V}'' \smallsetminus \mathbf{Z}_0$ obtained by the embedding $\mathbf{V}'' \smallsetminus \mathbf{Z}_0 \to \mathbf{V}'' \times \mathbb{A}^1$ given by $\phi^*(x_1)$.

For any $m', k \in \mathbb{N}$ we have

$$\lim_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_{m'}^{\mathbf{V}''\smallsetminus\mathbf{Z}_0,k} \cap B_{-1}^{\mathbf{V}'',k}(z)} |\tilde{\omega}| = 0$$

On the other hand, for any $m', k \in \mathbb{N}$ we have

$$\begin{aligned}
\int_{B_{m'}^{\mathbf{V}'' \smallsetminus \mathbf{Z}_0, k} \cap B_{-1}^{\mathbf{V}'', k}(z)} |\tilde{\omega}| &= \int_{\{x \in B_{m'}^{\mathbf{V}'', k} | val(x_1(\phi(x))) \leq m'\} \cap B_{-1}^{\mathbf{V}'', k}(z)} |\tilde{\omega}| \\
&= \int_{\{x \in B_{-1}^{\mathbf{V}'', k}(z) | val(x_1(\phi(x))) \leq m'\}} |\tilde{\omega}| \\
&= \int_{\{x \in B_{-1}^{\mathbf{V}'', k}(z) | val(x_1(\phi(x))) \leq m'\}} |g\tilde{\omega}| \\
&= \int_{(t\mathbb{F}_{\ell^k}[[t]] \smallsetminus t^{m'+1}\mathbb{F}_{\ell^k}[[t]]) \times (t\mathbb{F}_{\ell^k}[[t]])^{\dim \mathbf{X} - 1}} \left| \frac{dx_1 \wedge \cdots \wedge dx_2}{x_1^f} \right| \\
&= \left(\int_{(t\mathbb{F}_{\ell^k}[[t]] \smallsetminus t^{m'+1}\mathbb{F}_{\ell^k}[[t]])} \left| \frac{dx_1}{x_1^f} \right| \right) \cdot \mu^{\mathbb{A}^{\dim \mathbf{X} - 1}, k}(t\mathbb{F}_{\ell^k}[[t]])^{\dim \mathbf{X} - 1} \\
&= \frac{\ell^k - 1}{\ell^k} \left(\sum_{i=1}^{m'} \ell^{ik(f-1)} \right) \frac{1}{\ell^{k(\dim \mathbf{X} - 1)}} \\
&= \frac{\ell^k - 1}{\ell^k} \frac{\ell^{(m'+1)k(f-1)} - \ell^{k(f-1)}}{\ell^{k(f-1)} - 1} \frac{1}{\ell^{k(\dim \mathbf{X} - 1)}} \\
&= \ell^{k(f - \dim \mathbf{X} - 1)} \frac{(\ell^k - 1)(\ell^{m'k(f-1)} - 1)}{\ell^{k(f-1)} - 1}
\end{aligned}$$

So, for $m' > M + \dim \mathbf{X}$

$$\infty = \lim_{k \to \infty} \ell^{-kM} \ell^{k(f - \dim \mathbf{X} - 1)} \frac{(\ell^k - 1)(\ell^{m'k(f-1)} - 1)}{\ell^{k(f-1)} - 1} = \lim_{k \to \infty} \frac{1}{\ell^{kM}} \int_{B_n^{V' \smallsetminus Z_0, k} \cap B_{-1}^{V', k}} |\tilde{\omega}| = 0$$

Contradiction.

(2)⇒(3): Let $\mathbf{V}, \omega, k$ be as in Definition 9.0.1(3). Choose a rectification of $\mathbf{U} \cap \mathbf{V}$. WLOG assume $\mathbf{X} = \mathbf{V}$. Let $\phi : \tilde{\mathbf{X}} := \tilde{\mathbf{V}} \to \mathbf{V} = \mathbf{X}$ be a resolution of singularities as in Definition 9.0.1(2). Let $\tilde{\omega} := \phi^*(\omega)$. Let $\mathbf{Z}$ be the divisor of poles of $\tilde{\omega}$. Let $\tilde{\mathbf{U}} := \phi^{-1}(\mathbf{U})$. Choose a rectification on $\mathbf{U}$. We have to show that there exists M s.t. for any m we have

$$\int_{B_m^{\mathbf{U}, k} \cap \mathbf{X}(\mathbb{F}_{\ell^k}[[t]])} |\omega| < M(m+1)^M$$

By Lemma 4.2.5, this statement does not depend on the rectification on $\mathbf{U}$. By the valuative criterion we have $\tilde{\mathbf{X}}(\mathbb{F}_{\ell^k}[[t]]) = \phi^{-1}(\mathbf{X}(\mathbb{F}_{\ell^k}[[t]]))$. Here ϕ is interpreted as a map $\tilde{\mathbf{X}}(\mathbb{F}_{\ell^k}((t))) \to \mathbf{X}(\mathbb{F}_{\ell^k}((t)))$.

So it is enough to show that for some rectification of $\tilde{\mathbf{U}}$ there exists M s.t. for any m we have

$$\int_{B_m^{\tilde{\mathbf{U}}, k} \cap \tilde{\mathbf{X}}(\mathbb{F}_{\ell^k}[[t]])} |\tilde{\omega}| < M(m+1)^M$$

Let $\mathbf{W} \subset \tilde{\mathbf{X}}$ be the regular locus of $\tilde{\omega}$. We have $\tilde{\mathbf{U}} \subset \mathbf{W}$. Thus, for any rectification of $\mathbf{W}$ we can choose a rectification of $\tilde{\mathbf{U}}$ s.t. $B_m^{\tilde{\mathbf{U}},k} \subset B_m^{\mathbf{W},k}$ for any $m, k \in \mathbb{N}$. Therefore, it is enough to show that for some rectification of $\mathbf{W}$ there exists M s.t. for any m we have

$$\int_{B_m^{\mathbf{W},k} \cap \tilde{\mathbf{X}}(\mathbb{F}_{\ell^k}[[t]])} |\tilde{\omega}| < M(m+1)^M$$

Let $\tilde{\mathbf{X}} = \bigcup_{i=1}^I \mathbf{V}_i$ be an open affine cover such that there are:

- etale maps $\gamma_i : \mathbf{V}_i \to \mathbb{A}^d$
- monomial rational forms ω_i on $\mathbb{A}^d$ with powers ≥ -1, and
- regular functions $g \in O_{\tilde{\mathbf{X}}}(\mathbf{V}_i)$,

satisfying $\tilde{\omega}|_{\mathbf{V}_i} = g_i \gamma_i^*(\omega_i)$.

It is enough to show that for any i there is a rectification of $\mathbf{V}_i \cap \mathbf{U}$ and M s.t. for any m we have

$$\int_{B_m^{\mathbf{V}_i \cap \mathbf{W},k} \cap \tilde{\mathbf{V}}_i(\mathbb{F}_{\ell^k}[[t]])} |\gamma_i^*(\omega_i)| < M(m+1)^M$$

Let $\mathbf{W}_i$ be the regular locus of ω_i. This is a complement to coordinate hyperplanes in $\mathbb{A}^d$. So it is equipped with the standard embedding into $\mathbb{A}^{d+1}$ which gives us a simple rectification on $\mathbf{W}_i$. We can find a section on $\mathbf{V}_i \cap \mathbf{W}$ s.t. $B_m^{\mathbf{V}_i \cap \mathbf{W},k} \subset \gamma_i^{-1}(B_m^{\mathbf{W}_i,k})$ for any $m, k \in \mathbb{N}$.

Let $A_i \in \mathbb{N}$ s.t. for any field F and any $x \in F^d$ we have $\#\gamma_i^{-1}(x) < A$. For any k, m we obtain:

$$\int_{B_m^{\mathbf{V}_i \cap \mathbf{W},k} \cap \tilde{\mathbf{V}}_i(\mathbb{F}_{\ell^k}[[t]])} |\gamma_i^*(\omega_i)| \leq \int_{\gamma_i^{-1}(B_m^{\mathbf{W}_i,k} \cap \mathbb{F}_{\ell^k}[[t]]^d)} |\gamma_i^*(\omega_i)| \leq A \int_{B_m^{\mathbf{W}_i,k} \cap \mathbb{F}_{\ell^k}[[t]]^d} |\omega_i|$$

So, it is enough to show that for any i there is $M \in \mathbb{N}$ s.t. for any m we have

$$\int_{B_m^{\mathbf{W}_i,k} \cap \mathbb{F}_{\ell^k}[[t]]^d} |\omega_i| < M(m+1)^M.$$

This is a straightforward computation.

(3)⇒(2): The proof is similar to the implication (1)⇒(2) and we will not use this implication.

(2)⇒(1): The proof is similar to the implication (2)⇒(3) and we will not use this implication.

□

10. Almost integrability of $\underline{\mathfrak{g}}_i$

In this section we prove the following:

Theorem 10.0.1. *Let $i \in \mathbb{N}$ be an integer. Let $\mathbf{U}_i \subset \underline{\mathfrak{g}}_i$ be the preimage of $\underline{\mathfrak{c}}^{rss}$. Then $(\underline{\mathfrak{g}}_i, \mathbf{U}_i)$ is asymptotically almost integrable.*

Idea of the proof. We use the bound on the dimension of the jet schemes of $\underline{\mathfrak{g}}_i$ (see Theorem C). This bound gives us a bound on $\#\mathcal{J}_m(\underline{\mathfrak{g}}_i)(\mathbb{F}_{\ell^k})$ for large

k. This bounds the L^i norm of $p_*(\mu_0^{\underline{\mathfrak{g}},k}) * 1_{B_m^{\underline{\mathfrak{g}},k}}$. Since p is smooth over $\underline{\mathfrak{c}}^{rss}$, the measure $p_*(\mu_0^{\underline{\mathfrak{g}},k})$ is m-smooth in large balls in $\underline{\mathfrak{c}}^{rss}$. Thus we get a bound on the L^i norm of $p_*(\mu_0^{\underline{\mathfrak{g}},k})$ over large ball in $\underline{\mathfrak{c}}^{rss}$. This bounds the volume of a large ball in $\mathbf{U}_i$ as required. $\square$

Proof. Let c be as in Theorem C and let $M := c + 1$. Fix $m \in \mathbb{N}$. Let $\omega_{\underline{\mathfrak{g}}}$ be the standard top form on $\underline{\mathfrak{g}}$ (coming from the identification $\underline{\mathfrak{g}} = \mathbb{A}^{n^2}$) and $\omega_{\underline{\mathfrak{c}}}$ be the standard to form on $\underline{\mathfrak{c}}$ (coming from the identification $\underline{\mathfrak{c}} = \mathbb{A}^n$). Let $\omega_{\underline{\mathfrak{g}}_i} := \omega_{\underline{\mathfrak{g}}}^{\times_{\omega_{\underline{\mathfrak{c}}}} i}$. This is a rational top form on $\underline{\mathfrak{g}}_i$. Recall that $\underline{\mathfrak{g}}^r$ is the smooth locus of $p_0 : \underline{\mathfrak{g}} \to \underline{\mathfrak{c}}$. Thus $\omega_{\underline{\mathfrak{g}}_i}$ is regular and invertible on $(\underline{\mathfrak{g}}^r)^{\times_{\underline{\mathfrak{c}}} i}$. By Corollary 5.0.9, $\underline{\mathfrak{g}}^r$ is big in $\underline{\mathfrak{g}}$. Since p is flat (see Corollary 5.0.5), this implies that $(\underline{\mathfrak{g}}^r)^{\times_{\underline{\mathfrak{c}}} i}$ is big in $\underline{\mathfrak{g}}_i$. Therefore $\omega_{\underline{\mathfrak{g}}_i}$ is regular and invertible on $\underline{\mathfrak{g}}_i^{sm}$. Let $p^i : \underline{\mathfrak{g}}_i \to \underline{\mathfrak{c}}$ be the projection. Choose the standard μ-rectification of $\underline{\mathfrak{c}}$ and $\underline{\mathfrak{g}}$. Choose the simple μ-rectification on $\underline{\mathfrak{c}}^{rss}$ given by the embedding $\underline{\mathfrak{c}}^{rss} \to \underline{\mathfrak{c}} \times \mathbb{A}^1$ using the discriminant and the top-form induced from $\underline{\mathfrak{c}}$. This gives simple μ-rectifications on $\underline{\mathfrak{g}}^{rss}$ and $\mathbf{U}_i$.

By Lemma 4.2.11 there exists m' s.t. for any k there is an m'-smooth function f_k on $C^\infty(B_\infty^{\underline{\mathfrak{c}},k})$ s.t.

$$(p_0)_*(1_{B_0^{\underline{\mathfrak{g}},k}} \mu_m^{\underline{\mathfrak{g}}^{rss},k}) = f_k \mu_m^{\underline{\mathfrak{c}},k}.$$

We get

$$\begin{aligned}
\int_{B_m^{\mathbf{U}_i,k}\cap B_0^{\underline{\mathfrak{g}}_i,k}} |\omega_{\underline{\mathfrak{g}}_i}| &= \int_{B_m^{\underline{\mathfrak{c}}^{rss},k}\cap B_0^{\underline{\mathfrak{c}},k}} \left(\frac{p_*\left(|\omega_{\mathfrak{g}}|1_{B_0^{\underline{\mathfrak{g}},k}}\right)}{|\omega_{\mathfrak{c}}|}\right)^i |\omega_{\mathfrak{c}}| \\
&= \int_{B_m^{\underline{\mathfrak{c}}^{rss},k}\cap B_0^{\underline{\mathfrak{c}},k}} \left(\frac{p_*\left(1_{B_0^{\underline{\mathfrak{g}},k}}\mu_m^{\underline{\mathfrak{g}}^{rss},k}\right)}{|\omega_{\mathfrak{c}}|}\right)^i |\omega_{\mathfrak{c}}| \\
&= \int_{B_m^{\underline{\mathfrak{c}}^{rss},k}\cap B_0^{\underline{\mathfrak{c}},k}} (f_k)^i |\omega_{\mathfrak{c}}| \\
&= \frac{1}{|\omega_{\underline{\mathfrak{c}}}|(B_{-m'}^{\underline{\mathfrak{c}},k})} \int_{B_m^{\underline{\mathfrak{c}}^{rss},k}\cap B_0^{\underline{\mathfrak{c}},k}} \left(f_k * (|\omega_{\underline{\mathfrak{c}}}|1_{B_{-m'}^{\underline{\mathfrak{c}},k}})\right)^i |\omega_{\mathfrak{c}}| \\
&= \ell^{nm'k} \int_{B_m^{\underline{\mathfrak{c}}^{rss},k}\cap B_0^{\underline{\mathfrak{c}},k}} \left(f_k * (|\omega_{\underline{\mathfrak{c}}}|1_{B_{-m'}^{\underline{\mathfrak{c}},k}})\right)^i |\omega_{\mathfrak{c}}| \\
&= \ell^{nm'k} \int_{B_m^{\underline{\mathfrak{c}}^{rss},k}\cap B_0^{\underline{\mathfrak{c}},k}} \left(\frac{p_*\left(1_{B_0^{\underline{\mathfrak{g}},k}}\mu_m^{\underline{\mathfrak{g}}^{rss},k}\right)}{|\omega_{\mathfrak{c}}|} * (|\omega_{\underline{\mathfrak{c}}}|1_{B_{-m'}^{\underline{\mathfrak{c}},k}})\right)^i |\omega_{\mathfrak{c}}| \\
&= \ell^{nm'k} \int_{B_m^{\underline{\mathfrak{c}}^{rss},k}\cap B_0^{\underline{\mathfrak{c}},k}} \left(p_*\left(1_{B_0^{\underline{\mathfrak{g}},k}}\mu_m^{\underline{\mathfrak{g}}^{rss},k}\right) * (1_{B_{-m'}^{\underline{\mathfrak{c}},k}})\right)^i |\omega_{\mathfrak{c}}| \\
&= \ell^{nm'k} \int_{B_m^{\underline{\mathfrak{c}}^{rss},k}\cap B_0^{\underline{\mathfrak{c}},k}} \left(p_*\left(|\omega_{\mathfrak{g}}|1_{B_0^{\underline{\mathfrak{g}},k}}\right) * (1_{B_{-m'}^{\underline{\mathfrak{c}},k}})\right)^i |\omega_{\mathfrak{c}}| \\
&< \ell^{nm'k} \int_{B_0^{\underline{\mathfrak{c}},k}} \left(p_*\left(|\omega_{\mathfrak{g}}|1_{B_0^{\underline{\mathfrak{g}},k}}\right) * (1_{B_{-m'}^{\underline{\mathfrak{c}},k}})\right)^i |\omega_{\mathfrak{c}}| \\
&= \ell^{nm'k} \sum_{B\in B_0^{\underline{\mathfrak{c}},k}/B_{-m'}^{\underline{\mathfrak{c}},k}} \int_B \left(p_*\left(|\omega_{\mathfrak{g}}|1_{B_0^{\underline{\mathfrak{g}},k}}\right) * (1_{B_{-m'}^{\underline{\mathfrak{c}},k}})\right)^i |\omega_{\mathfrak{c}}| \\
&= \ell^{nm'k} \sum_{B\in B_0^{\underline{\mathfrak{c}},k}/B_{-m'}^{\underline{\mathfrak{c}},k}} \left(\frac{|\omega_{\underline{\mathfrak{g}}}|(p^{-1}(B)\cap B_0^{\underline{\mathfrak{g}},k})}{|\omega_{\underline{\mathfrak{c}}}|(B)}\right)^i |\omega_{\underline{\mathfrak{c}}}|(B) \\
&= \ell^{(i-1)nm'k} \sum_{B\in B_0^{\underline{\mathfrak{c}},k}/B_{-m'}^{\underline{\mathfrak{c}},k}} \left(|\omega_{\underline{\mathfrak{g}}}|(p^{-1}(B)\cap B_0^{\underline{\mathfrak{g}},k})\right)^i \\
&= \ell^{(i-1)nm'k} \sum_{x\in\mathcal{J}_{m'}(\underline{\mathfrak{c}})(\mathbb{F}_{\ell^k})} \left(\#\left((\mathcal{J}_{m'}(p))^{-1}(x)\right)|\omega_{\mathfrak{g}}|(B_{-m'}^{\underline{\mathfrak{g}},k})\right)^i \\
&= \ell^{(i-1)nm'k-im'n^2k} \sum_{x\in\mathcal{J}_{m'}(\underline{\mathfrak{c}})(\mathbb{F}_{\ell^k})} \left(\#\left((\mathcal{J}_{m'}(p))^{-1}(x)\right)\right)^i \\
&= \ell^{(i-1)nm'k-im'n^2k}\#\mathcal{J}_{m'}(\underline{\mathfrak{g}}_i)(\mathbb{F}_{\ell^k}) = \ell^{-m'k\dim\underline{\mathfrak{g}}_i}\#\mathcal{J}_{m'}(\underline{\mathfrak{g}}_i)(\mathbb{F}_{\ell^k}).
\end{aligned}$$

Therefore

$$
\begin{aligned}
0 \le \lim_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_m^{\mathbf{U}_i,k} \cap B_0^{\underline{\mathfrak{g}}_i,k}} |\omega_{\underline{\mathfrak{g}}_i}| &\le \lim_{k\to\infty} \frac{\ell^{-m'k \dim \underline{\mathfrak{g}}_i} \# \mathcal{J}_{m'}(\underline{\mathfrak{g}}_i)(\mathbb{F}_{\ell^k})}{\ell^{kM}} \\
= \lim_{k\to\infty} \frac{\# \mathcal{J}_{m'}(\underline{\mathfrak{g}}_i)(\mathbb{F}_{\ell^k})}{\ell^{k(M+m' \dim \underline{\mathfrak{g}}_i)}} &= \lim_{k\to\infty} \frac{\# \mathcal{J}_{m'}(\underline{\mathfrak{g}}_i)(\mathbb{F}_{\ell^k})}{\ell^{k(c+1+m' \dim \underline{\mathfrak{g}}_i)}} \\
\le \lim_{k\to\infty} \frac{\# \mathcal{J}_{m'}(\underline{\mathfrak{g}}_i)(\mathbb{F}_{\ell^k})}{\ell^{k(1+\dim \mathcal{J}_{m'}(\underline{\mathfrak{g}}_i))}} &= \lim_{k\to\infty} \ell^{-k} \frac{\# \mathcal{J}_{m'}(\underline{\mathfrak{g}}_i)(\mathbb{F}_{\ell^k})}{\ell^{k(\dim \mathcal{J}_{m'}(\underline{\mathfrak{g}}_i))}} = 0.
\end{aligned}
$$

The last equality follows from the Lang-Weil bound. So

$$
\lim_{k\to\infty} \frac{1}{\ell^{kM}} \int_{B_m^{\mathbf{U}_i,k} \cap B_0^{\underline{\mathfrak{g}}_i,k}} |\omega_{\underline{\mathfrak{g}}_i}| = 0,
$$

as required. □

11. Proof of Theorem D

We will need the following:

Lemma 11.0.1. *Let $\phi : \mathbf{X} \to \mathbf{Y}$ be a flat map of smooth (μ-rectified) varieties. Assume that the smooth locus of ϕ is big in $\mathbf{X}$. Let $\mathbf{U} \subset \mathbf{Y}$ be an open dense subset of the locus of regular values of ϕ. Let $i, k \in \mathbb{N}$. Let $\phi_i : \mathbf{X}_i := \mathbf{X}^{\times_{\mathbf{Y}} i} \to \mathbf{Y}$ be the projection. Let $\mathbf{V} = \phi_i^{-1}(\mathbf{U})$.*

Assume that $(\mathbf{X}_i, \mathbf{V})$ is analytically almost integrable.

Then $\phi_*(\mu_0^{\mathbf{X},k}) \in L^{i'}(B_\infty^{\mathbf{Y},k})$ for any $i' < i$.

Proof. WLOG assume that the μ-rectifications of $\mathbf{X}$ and $\mathbf{Y}$ are simple.

Step 1. $\exists M > 0$ s.t. $\forall m \in \mathbb{N}$ we have $\int_{B_m^{\mathbf{U},k} \cap B_0^{\mathbf{Y},k}} \left(\frac{\phi_*(\mu_0^{\mathbf{X},k})}{\mu^{\mathbf{Y},k}} \right)^i \mu^{\mathbf{Y},k} < M m^M$.

Using the embedding $\mathbf{X}_i \to \mathbf{X}^i$ we obtain an embedding of $\mathbf{X}_i$ into an affine space. Let $\omega_{\mathbf{X}}$, $\omega_{\mathbf{Y}}$ be the forms on $\mathbf{X}$ and $\mathbf{Y}$ and let $\omega_{\mathbf{X}_i} := \omega_{\mathbf{X}}^{\times_{\omega_{\mathbf{Y}}} i}$. This is a rational form. The condition implies that it is regular on a big subset of $\mathbf{X}_i$ and hence can be extended to the smooth locus of $\mathbf{X}_i$. This together with the embedding above gives us a μ-rectification of $\mathbf{X}_i$. Choose a μ-rectification on $\mathbf{U}$, and choose a rectification of $\mathbf{V}$ s.t. for any $m \in \mathbb{N}$ we have

$$
B_m^{\mathbf{V},k} = B_m^{\mathbf{X}_i,k} \cap \phi_i^{-1}(B_m^{\mathbf{U},k}).
$$

Since $(\mathbf{X}_i, \mathbf{V})$ is analytically almost integrable we can find a constant $M \in \mathbb{N}$ s.t. for any $m \in \mathbb{N}$ we have

$$
\int_{B_m^{\mathbf{V},k} \cap B_0^{\mathbf{X}_i,k}} |\omega_{\mathbf{X}_i}| < M m^M.
$$

Let $m \in \mathbb{N}$. We obtain:

$$\begin{aligned}\int_{B_m^{\mathbf{U},k}\cap B_0^{\mathbf{Y},k}} \left(\frac{\phi_*(\mu_0^{\mathbf{X},k})}{\mu^{\mathbf{Y},k}}\right)^i \mu^{\mathbf{Y},k} &= \int_{\phi_i^{-1}(B_m^{\mathbf{U},k}\cap B_0^{\mathbf{Y},k})\cap B_0^{\mathbf{X}_i,k}} |\omega_{\mathbf{X}_i}| \\ &= \int_{\phi_i^{-1}(B_m^{\mathbf{U},k})\cap\phi_i^{-1}(B_0^{\mathbf{Y},k})\cap B_0^{\mathbf{X}_i,k}} |\omega_{\mathbf{X}_i}| \\ &= \int_{(B_m^{\mathbf{V},k})\cap\phi^{-1}(B_0^{\mathbf{Y},k})\cap B_0^{\mathbf{X}_i,k}} |\omega_{\mathbf{X}_i}| \\ &= \int_{B_m^{\mathbf{V},k}\cap B_0^{\mathbf{X}_i,k}} |\omega_{\mathbf{X}_i}| < Mm^M\end{aligned}$$

Step 2. there is $0 < g \in L^{<\infty}_{\text{loc}}(B_\infty^{\mathbf{Y},k})$ such that $\frac{\phi_*(\mu_0^{\mathbf{X},k})}{g\mu^{\mathbf{Y},k}} \in L^i(B_\infty^{\mathbf{Y},k})$. Take

$$g(y) := \begin{cases}\min\{m \geq 1 | y \in B_m^{\mathbf{U},k}\}^{M+2}, & \text{if } y \in B_\infty^{\mathbf{U},k} \\ 0, & \text{otherwise}\end{cases}$$

For any $m \geq 1$ we have:

$$\begin{aligned}\int_{(B_m^{\mathbf{U},k}\setminus B_{m-1}^{\mathbf{U},k})\cap B_0^{\mathbf{Y},k}} \left(\frac{\phi_*(\mu_0^{\mathbf{X},k})}{g\mu^{\mathbf{Y},k}}\right)^i \mu_Y &= \int_{(B_m^{\mathbf{U},k}\setminus B_{m-1}^{\mathbf{U},k})\cap B_0^{\mathbf{Y},k}} \left(\frac{\phi_*(\mu_0^{\mathbf{X},k})}{m^{M+2}\mu^{\mathbf{Y},k}}\right)^i \mu_Y \\ &= \frac{1}{m^{(M+2)i}} \int_{(B_m^{\mathbf{U},k}\setminus B_{m-1}^{\mathbf{U},k})\cap B_0^{\mathbf{Y},k}} \left(\frac{\phi_*(\mu_0^{\mathbf{X},k})}{\mu^{\mathbf{Y},k}}\right)^i \\ &< \frac{1}{m^{(M+2)i}} \int_{(B_m^{\mathbf{U},k})\cap B_0^{\mathbf{Y},k}} \left(\frac{\phi_*(\mu_0^{\mathbf{X},k})}{\mu^{\mathbf{Y},k}}\right)^i \\ &< \frac{Mm^M}{m^{(M+2)k}} \leq \frac{M}{m^2}\end{aligned}$$

Thus

$$\begin{aligned}\int_{B_\infty^{\mathbf{Y},k}} \left(\frac{\mu_0^{\mathbf{X},k}}{g\mu^{\mathbf{Y},k}}\right)^i \mu_{\mathbf{Y}} &= \int_{B_\infty^{\mathbf{U},k}} \left(\frac{\mu_0^{\mathbf{X},k}}{g\mu^{\mathbf{Y},k}}\right)^i \mu_{\mathbf{Y}} \\ &= \int_{B_0^{\mathbf{U},k}} \left(\frac{\mu_0^{\mathbf{X},k}}{g\mu^{\mathbf{Y},k}}\right)^i \mu_{\mathbf{Y}} + \sum_{m=1}^{\infty} \int_{B_m^{\mathbf{U},k}\setminus B_{m-1}^{\mathbf{U},k}} \left(\frac{\mu_0^{\mathbf{X},k}}{g\mu^{\mathbf{Y},k}}\right)^i \mu_{\mathbf{Y}} \\ &< M + \sum_{m=1}^{\infty} \frac{M}{m^2} < 3M.\end{aligned}$$

It remains to show that $g \in L^{<\infty}_{\text{loc}}(B_\infty^{\mathbf{Y},k})$. It is easy to see that $g|_{B_\infty^{\mathbf{U},k}}$ is a norm function as described in §3. Thus by Proposition 3.0.1 $g \in L^{<\infty}_{\text{loc}}(B_\infty^{\mathbf{Y},k})$.

Step 3. End of the proof.

Let $i' < i$ and let $A > 1$ be such that $\frac{1}{i} + \frac{1}{A} = \frac{1}{i'}$. By the previous step $g \in L^A_{\text{loc}}(B_\infty^{\mathbf{Y},k})$ and $\frac{\phi_*(\mu_0^{\mathbf{X},k})}{g\mu^{\mathbf{Y},k}} \in L^i_{\text{loc}}(B_\infty^{\mathbf{Y},k})$.

Let $m \in \mathbb{Z}_{>0}$. We obtain:

- $g 1_{B_m^{\mathbf{Y},k}} \in L^A(B_m^{\mathbf{Y},k})$
- $\frac{\phi_*(\mu_0^{\mathbf{X},k})}{g\mu^{\mathbf{Y},k}} 1_{B_m^{\mathbf{Y},k}} \in L^i(B_m^{\mathbf{Y},k})$.

We recall the generalized Hölder inequality. Suppose that:

- r_1, r_2, r are positive and satisfy $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$
- $f_1 \in L^{r_1}(B_m^{\mathbf{Y},k}), f_2 \in L^{r_2}(B_m^{\mathbf{Y},k})$.

Then

$$||f_1 f_2||_r \leq ||f_1||_{r_1} ||f_2||_{r_2}$$

See [WZ15, Chapter 8, Exercise 6].

Thus we have:

$$||1_{B_m^{\mathbf{Y},k}} \frac{\phi_*(\mu_0^{\mathbf{X},k})}{\mu^{\mathbf{Y},k}}||_{i'} = ||1_{B_m^{\mathbf{Y},k}} g \frac{\phi_*(\mu_0^{\mathbf{X},k})}{g\mu^{\mathbf{Y},k}}||_{i'} \leq ||1_{B_m^{\mathbf{Y},k}} g||_A ||1_{B_m^{\mathbf{Y},k}} \frac{\phi_*(\mu_0^{\mathbf{X},k})}{g\mu^{\mathbf{Y},k}}||_i < \infty,$$

as requested.

□

Proof of Theorem D. Using the homothety action of $\mathbb{G}_m$ on $\underline{\mathfrak{g}}$ we can assume, WLOG, that $\mu \leq c\mu_0^{\underline{\mathfrak{g}},1}$ for some constant $c > 0$. Hence we can assume, WLOG, that $\mu = \mu_0^{\underline{\mathfrak{g}},1}$.

Let $\mathbf{U}_i \subset \underline{\mathfrak{g}}_i$ be the preimage of $\underline{\mathfrak{c}}^{rss}$ as in Theorem 10.0.1.

By Theorem 10.0.1 $(\underline{\mathfrak{g}}_i, \mathbf{U}_i)$ is asymptotically almost integrable. By Theorem 9.0.3, this together with the assumption implies that $(\underline{\mathfrak{g}}_i, \mathbf{U}_i)$ is analytically almost integrable. By Lemma 11.0.1, applied to the map $p : \underline{\mathfrak{g}} \to \underline{\mathfrak{c}}$, this implies the assertion. Note that the rest of the properties of p required by Lemma 11.0.1 are verified in §5. □

12. Alternative versions of Theorem D

In Theorem D one can replace the condition of existence of strong resolution of $\underline{\mathfrak{g}}_i$ with one of the following 2 conditions:

(1) the defining ideal of $\underline{\mathfrak{g}}_i$ inside $\underline{\mathfrak{g}}^{\times i}$ has monomial principalization (see Definition 12.0.1 below).

(2) (a) The defining ideal of $\mathbf{N}$ inside $\underline{\mathfrak{g}}$ has (monomial) principalization
    (b) $\underline{\mathfrak{g}}_i$ has a resolution (not necessarily strong).

Definition 12.0.1 (Monomial Principalization)**.**

(1) An ideal sheaf I on a smooth variety $\mathbf{X}$ is called locally monomially principal if there exist

- *a finite Zariski open affine cover $\mathbf{X} = \bigcup \mathbf{U}_i$,*
- *a collection of etale maps $\phi_i : \mathbf{U}_i \to \mathbb{A}^{d_i}$, and*
- *monomials f_i on $\mathbb{A}^{d_i}$*

such that for any i we have $I|_{\mathbf{U}_i} = \phi_i^*(f_i) O_{\mathbf{U}_i}$.

(2) *Given an ideal sheaf I on a smooth variety $\mathbf{X}$, we say that $\phi : \tilde{\mathbf{X}} \to \mathbf{X}$ is a monomial principalization of I if it satisfies:*
   - *$\tilde{\mathbf{X}}$ is smooth and ϕ is a proper birational map (a modification).*
   - *The pulled-back ideal sheaf $\phi^*(I)$ is locally monomially principal.*

Since the difference between these 2 versions of Theorem D is technical and it is not clear how useful it is going to be, we will not give complete proofs of these versions but only a sketch.

12.1. **Sketch of the proof of Theorem D with condition of strong resolution replaced with Condition (1).** The proof is based on the following version of [Mus01, Theorem 3.1]

Theorem 12.1.1. *Let $\mathbf{X} \subset \mathbf{M}$ be a closed subvariety of a smooth algebraic variety. Let $\gamma : \tilde{\mathbf{M}} \to \mathbf{M}$ be a principalization of the defining ideal sheaf $\mathcal{I}_{\mathbf{X}}$ of $\mathbf{X}$ inside $\mathbf{M}$ Let a_i, b_i be as in* [Mus01, Theorem 3.1]*. That is a_i are the coefficients of the components of the pre-image of $\mathcal{I}_{\mathbf{X}}$ and b_i are the coefficients of the components of the discrepancy divisor of γ.*

Then, the following statements are equivalent:

(i) *For every $i \geq 1$, we have $b_i \geq (\dim(\mathbf{M}) - \dim(\mathbf{X}))a_i - 1$.*
(ii) *There is a constant c s.t. $\dim \mathcal{J}_m(\mathbf{X}) \leq (m+1)\dim \mathbf{X} + c$, for every $m \geq 1$.*
(iii) *There exists $M \in \mathbb{N}$ s.t. for any $m, k \in \mathbb{N}$ we have*

$$\#\mathcal{J}_m(\mathbf{X})(\mathbb{F}_{\ell^k}) < M m^M \ell^{km \dim \mathbf{X}}.$$

The proof of this theorem is parallel to the original proof of [Mus01, Theorem 3.1]. One also has to use the Denef's formula, but one can use the usual Denef formula and not the motivic one.

Similarly to the proof of Theorem 10.0.1, this theorem implies that, under assumption (1), the variety $\underline{\mathfrak{g}}_i$ is analytically almost integrable.

The rest of the proof is the same as the proof of Theorem D.

13. Proof of Theorem E

13.1. **an-FRS maps of analytic varieties.** Let $F := \mathbb{F}_\ell((t))$. Note that we can extend the notion of an-FRS morphisms (see Definition 1.3.5 (2)) to the set-up of morphisms between F-analytic varieties in a natural way.

We require the following straightforward lemma about properties of an-FRS maps.

Lemma 13.1.1. *Let $\gamma : X \to Y$ be a morphism of smooth F-analytic varieties. Let $\delta_i : \tilde{X}_i \to X$ for $i \in I$ be a (possibly infinite) collection of submersions. Then:*

(1) *if γ is an-FRS then, for each i, $\gamma \circ \delta_i$ is an-FRS.*
(2) *if $\bigcup_{i \in I} \delta_i(\tilde{X}_i) = X$ and $\gamma \circ \delta_i$ is an-FRS for all i, then γ is an-FRS.*

13.2. **Harish-Chandra descent for an-FRS maps.**

Notation 13.2.1. *Denote*

- $\mathfrak{g} := \underline{\mathfrak{g}}(F)$
- $\mathfrak{g}^{ss}$:= *the collection of semi-simple elements in* $\mathfrak{g}$*, i.e. those elements with a separable minimal polynomial.*
- $\mathbf{N}_{insep} \subset \underline{\mathfrak{g}}$ *- be the locus of all matrices whose characteristic polynomial is totally inseparable (i.e. has a single root over the algebraic closure). Note that it has a natural structure of an algebraic variety.*
- $N_{insep} := \mathbf{N}_{insep}(F)$
- $\mathbf{N}_{insep,s} := \mathbf{N}_{insep} \smallsetminus \underline{\mathfrak{g}}^r$ *where* $\underline{\mathfrak{g}}^r$ *is the smooth locus of* p*.*
- $G := \mathbf{G}(F)$
- *For* $x \in \mathfrak{g}^{ss}$ *denote by* $\alpha_x : G \times \mathfrak{g}_x \to \mathfrak{g}$ *be the action map*
- *For a map* $\alpha : X \to Y$ *of smooth* F*-analytic varieties denote by* $\alpha^{reg} \subset X$ *to be the collection of regular points.*
- $\mathfrak{z} := \underline{\mathfrak{z}}(F)$
- $N := \mathbf{N}(F)$

Lemma 13.2.2. $\bigcup_{x \in \mathfrak{g}^{ss} \smallsetminus \mathfrak{z}} \alpha_x(\alpha_x^{reg}) = \mathfrak{g} \smallsetminus N_{insep}$.

Proof. Let $A := \bigcup_{x \in \mathfrak{g}^{ss} \smallsetminus \mathfrak{z}} \alpha_x(\alpha_x^{reg})$ take $y \in \mathfrak{g} \smallsetminus N_{insep}$ we have to show that $y \in A$.

Case 1: The minimal polynomial of y is irreducible.
Let f be the minimal polynomial of y. we can write $f(s) = g(s^{p^k})$ where g is separable polynomial. Let $x := y^{p^k}$. It is left to show that $(1, y)$ is a regular point of α_x. In other words we have to show that $\mathfrak{g}_x + [\mathfrak{g}, y] = \mathfrak{g}$. Passing to the orthogonal compliment w.r.t. the trace form, we need to show that $\mathfrak{g}_x^{\perp} \cap [\mathfrak{g}, y]^{\perp} = 0$. Now:

$$\mathfrak{g}_x^{\perp} \cap [\mathfrak{g}, y]^{\perp} = [\mathfrak{g}, x] \cap \mathfrak{g}_y \subset [\mathfrak{g}, x] \cap \mathfrak{g}_x = 0$$

Case 2: The minimal polynomial of y is a power of an irreducible polynomial.
Let $f = g^k$ be the minimal polynomial of y, with g being irreducible. We note that using rational canonical form we can find $x \in \mathfrak{g}$ s.t. $\overline{Ad(G) \cdot y} \ni x$ and the minimal polynomial of x is g. By the previous case, $x \in A$. Since A is open we are done.

Case 3: The minimal polynomial of y is product of 2 co-prime polynomials.
In this case we can use the Primary decomposition theorem (from linear algebra) to find $x \in \mathfrak{g}^{ss} \smallsetminus \mathfrak{z}$ s.t. $\mathfrak{g}_x \supset \mathfrak{g}_y$. Now the claim is proven as in Case 1.

∎

Lemmas 13.2.2 and 13.1.1 give us:

Corollary 13.2.3. *Assume that Theorem E holds for any smaller value of* n*. Then* $p|_{\underline{\mathfrak{g}} \smallsetminus \mathbf{N}_{insep}}$ *is an-FRS.*

13.3. **Slices to nilpotent orbits.** The following Lemma is a version of Lemma 6.2.1.

Lemma 13.3.1. *Assume* $\mathrm{char}(\mathbb{F}_\ell) > \frac{n}{2}$. *Let* $x \in \mathbf{N}(\mathbb{F}_\ell)$ *be a non-regular element. Then there exist*

- *a linear subspace* $\mathbf{M} \subset \underline{\mathfrak{g}}$, *and*
- *a positive* $\mathbb{G}_m$ *action on* $\mathbf{M}$,

such that

(1) *the action map* $\mathbf{G} \times (x + \mathbf{M}) \to \underline{\mathfrak{g}}$ *is a submersion.*
(2) *for any nilpotent* $y \in x + (\mathbf{M}(\bar{\mathbb{F}}_\ell) \smallsetminus 0)$ *we have* $\dim \mathbf{G}_{\bar{\mathbb{F}}_\ell} \cdot y > \dim \mathbf{G}_{\bar{\mathbb{F}}_\ell} \cdot x$
(3) *The Chevalley map* $p|_{x+\mathbf{M}}$ *intertwines the* $\mathbb{G}_m$ *action on* $x + \mathbf{M}$ *(given by the identification* $y \mapsto x + y$ *between* $\mathbf{M}$ *and* $x + \mathbf{M}$*) with the* $\mathbb{G}_m$ *action on* $\underline{\mathfrak{c}}$ *given by* $(\lambda \cdot f)(y) := \lambda^n f(\lambda^{-1} \cdot y)$.
(4) *If in addition* x *is not subregular, then the sum of the exponents of the* $\mathbb{G}_m$ *action on* $\mathbf{M}$ *is larger than* $\frac{n(n+1)}{2} + 1$.
(5) $\underline{\mathfrak{z}} \subset \mathbf{M}$ *is* $\mathbb{G}_m$ *invariant and the exponent of the action of* $\mathbb{G}_m$ *on* $\underline{\mathfrak{z}}$ *is* 1.

The proof is analogous to the proof of Lemma 6.2.1. We will start with some preparations.

Notation 13.3.2. *Let* x *be a nilpotent element in Jordan canonical form. Let* $\mathbf{L}_x$ *be the slice defined in Notation 6.2.3. Denote*

(1) $\mathbf{M}_x^0 := \{\{x_{ij}\} \in \mathbf{L}_x | x_{nn} = 0\}$.
(2) $\mathbf{M}_x := \mathbf{M}_x^0 + \underline{\mathfrak{z}}$.

Proof of Lemma 13.3.1. WLOG we may assume that x is in a Jordan form and the size of the largest block is smaller than $\mathrm{char}(\mathbb{F}_\ell)$.

Take $\mathbf{M} := \mathbf{M}_x$. Define the action of $\mathbb{G}_m$ on $\mathbf{M}$ by

$$\lambda \star A := \phi_x(\lambda) \cdot A.$$

where ϕ_x is the morphism defined in Notation 6.2.3(7) and $\cdot$ is the action described in Notation 6.2.3(1).

It is easy to see that this is a positive action.

Conditions (1,2,4) are proven in the same way as in Lemma 6.2.1.

Conditions (3,5) are evident.

□

13.4. **Proof of Theorem E.** We prove the theorem by induction n. Throughout the section we assume the validity of the result for smaller values of n.

The following is obvious:

Lemma 13.4.1. $p|_{\underline{\mathfrak{g}}^r}$ *is an-FRS.*

We obtain:

Corollary 13.4.2. *Let* $\mathbf{N}_{insep,s} := \mathbf{N}_{insep} \smallsetminus \underline{\mathfrak{g}}^r$. *Then* $p|_{\underline{\mathfrak{g}} \smallsetminus \mathbf{N}_{insep,s}}$ *is an-FRS.*

Remark 13.4.3. *Assume* $\mathrm{char}(\mathbb{F}_\ell) > \frac{n}{2}$. *Then it is easy to see that* $N_{insep,s} \subset N + \mathfrak{z}$.

The following is a global non-uniform version of Lemma 6.1.1.

Lemma 13.4.4. *Let $\gamma : \mathbb{A}^I \to \mathbb{A}^J$ be a polynomial map. Assume that:*

(1) we are given positive actions of $\mathbb{G}_m$ on $\mathbb{A}^I$ and $\mathbb{A}^J$ by

$$s \cdot (x_1, \ldots, x_I) = (s^{\lambda_1} x_1, \ldots, s^{\lambda_I} x_I)$$

and

$$s \cdot (y_1, \ldots, y_J) = (s^{\mu_1} y_1, \ldots, s^{\mu_J} y_J)$$

respectively.

(2) we are given an action $\star$ of $\mathbb{G}_a$ on $\mathbb{A}^J$ s.t. for any $z \in \mathbb{G}_a(\bar{\mathbb{F}}_\ell)$ and $x \in \mathbb{A}^I(\bar{\mathbb{F}}_\ell)$ we have $\gamma(x+z) = z \star \gamma(x)$. Here $\mathbb{G}_a$ embeds into $\mathbb{A}^I$ as the first coordinate.

(3) $\star$ preserves the Haar measure on F^I.

(4) $\sum_i \lambda_i > 1 + \sum_i \mu_i$.

(5) $\lambda_1 = 1$

(6) $\gamma|_{\mathbb{A}^I \smallsetminus \mathbb{A}^1} : (\mathbb{A}^I \smallsetminus \mathbb{A}^1) \to \mathbb{A}^J$ is an-FRS.

Then γ is an-FRS.

Proof. WLOG we may assume that μ is an Haar measure on a ball $B \subset \mathbb{F}_\ell((t))^I$ around the origin. Using the $\mathbb{G}_m$ action we may assume WLOG that B is the unit ball $\mathbb{F}_\ell[[t]]^I$.

Define another action of $\mathbb{G}_m$ on $\mathbb{A}^I$ by:

$$s * (x_1, \ldots, x_I) = (x_1, s^{\lambda_2} x_2, \ldots, s^{\lambda_I} x_I).$$

For $i \in \mathbb{N}$, define $C_i := t^i * B \smallsetminus t^{i+1} * B$ and $\nu_i := \mu 1_{C_i}$.

Let $S_i \subset \mathbb{F}_\ell[[t]]$ be the collection of polynomials of degree $\leq i-1$. We will also consider S_i as a subset of $\mathbb{F}_\ell((t))^I$ using the embedding $\mathbb{A}^1 \subset \mathbb{A}^I$ corresponding to the first coordinate. Note that

$$C_i = \bigcup_{s \in S_i} \left(s + t^i \cdot C_0 \right).$$

This implies

$$\nu_i = \ell^{-i \sum_{j=1}^{I} \lambda_j} \sum_{s \in S_i} \left(sh_s(t^i \cdot \nu_0) \right),$$

where sh_s stands for shift of a measure by s.

Denote

$$g := \frac{\gamma_*(\nu_0)}{\mu^{\mathbb{A}^J,1}}$$

and let

$$M = ||g||_\infty.$$

We obtain:

$$\begin{aligned}\gamma_*(\mu) =& \sum_{i=0}^{\infty}\gamma_*(\nu_i) = \sum_{i=0}^{\infty}\ell^{-i\sum_{j=1}^{I}\lambda_j}\sum_{s\in S_i}\gamma_*\left(sh_s(t^i\cdot\nu_0)\right) = \sum_{i=0}^{\infty}\ell^{-i\sum_{j=1}^{I}\lambda_j}\sum_{s\in S_i}s\star\gamma_*\left(t^i\cdot\nu_0\right) = \\ =&\sum_{i=0}^{\infty}\ell^{-i\sum_{j=1}^{I}\lambda_j}\sum_{s\in S_i}s\star\left(t^i\cdot\gamma_*\left(\nu_0\right)\right) = \sum_{i=0}^{\infty}\ell^{-i\sum_{j=1}^{I}\lambda_j}\sum_{s\in S_i}s\star\left(t^i\cdot\left(\frac{\gamma_*\left(\nu_0\right)}{\mu^{\mathbb{A}^J,1}}\mu^{\mathbb{A}^J,1}\right)\right) = \\ =&\sum_{i=0}^{\infty}\ell^{-i\sum_{j=1}^{I}\lambda_j}\sum_{s\in S_i}s\star\left(\left(t^i\cdot g\right)\left(t^i\cdot\mu^{\mathbb{A}^J,1}\right)\right) = \\ =&\sum_{i=0}^{\infty}\ell^{-i\sum_{j=1}^{I}\lambda_j+i\sum_{j=1}^{J}\mu_j}\sum_{s\in S_i}s\star\left(\left(t^i\cdot g\right)\mu^{\mathbb{A}^J,1}\right) \leq \\ \leq&\sum_{i=0}^{\infty}\ell^{-2i}\sum_{s\in S_i}s\star\left(M\mu^{\mathbb{A}^J,1}\right) = M\sum_{i=0}^{\infty}\ell^{-2i}\sum_{s\in S_i}\left(\mu^{\mathbb{A}^J,1}\right) = M\sum_{i=0}^{\infty}\ell^{-2i}|S_i|\mu^{\mathbb{A}^J,1} = \\ =&M\sum_{i=0}^{\infty}\ell^{-i}\mu^{\mathbb{A}^J,1} \leq 2M\mu^{\mathbb{A}^J,1}.\end{aligned}$$

This implies the assertion. □

Lemma 13.4.5. *Assume* $\operatorname{char}\mathbb{F}_\ell > n/2$. *Let* $\mathbf{O} \subset \mathbf{N}$ *be a non-regular nilpotent orbit. Then for any field* $E/\mathbb{F}_\ell$ *we have*

$$(\underline{\mathfrak{z}} + \mathbf{O})(E) = \underline{\mathfrak{z}}(E) + \mathbf{O}(E).$$

Proof. Let $x \in (\underline{\mathfrak{z}} + \mathbf{O})(E)$. First note that it is non-regular as an element in $x \in \underline{\mathfrak{g}}(\bar{E})$. This implies that it is non-regular as an element in $\underline{\mathfrak{g}}(E)$. Let f be its characteristic polynomial and g be its minimal polynomial. Let $h = f/g$. Note that both h, g are defined over E. Also, for some $\lambda \in \bar{E}$ and $k \in \mathbb{N}$ we can write: $g = (x-\lambda)^k$ and $h = (x-\lambda)^{n-k}$. Let $m = \min(k, n-k)$. We get that $(x-\lambda)^m$ is defined over E and that $0 < m < p$. This implies that $\lambda \in E$, which implies the assertion. □

Lemma 13.1.1 implies the following:

Corollary 13.4.6. *Let* $\gamma : \mathbf{X} \to \mathbf{Y}$ *be a morphism of algebraic varieties. Let* $\delta : \tilde{\mathbf{X}} \to \mathbf{X}$ *be a submersion. Then:*

(1) if γ *is an-FRS then so is* $\gamma \circ \delta$.
(2) If $\delta(\tilde{\mathbf{X}}(\mathbb{F}_\ell((t)))) = \mathbf{X}(\mathbb{F}_\ell((t)))$ *and* $\gamma \circ \delta$ *is an-FRS then so is* γ.

The classical argument for the bounds of Cauchy and Lagrange gives:

Lemma 13.4.7. *Let* $f \in \underline{\mathfrak{c}}(O_{\bar{F}})$. *Then any root* $\lambda \in \bar{F}$ *of* f *satisfies* $\lambda \in O_{\bar{F}}$.

Lemma 13.4.8. *Let* $f \in O_{\bar{F}}[x]$ *be a monic polynomial in one variable with coefficients in* $O_{\bar{F}}$. *Then*

$$\int_{O_F} val(f(x))dx \leq \frac{\deg(f)}{\ell - 1}.$$

Proof.
Case 1: $f(x) = x$

$$\int_{O_F} val(x)dx = \sum_{m=0}^{\infty} \int_{|x|=\ell^{-m}} val(x) = \sum_{m=0}^{\infty} mVol(\{x \in O_F : |x| = \ell^{-m}\})dx = C \sum_{m=0}^{\infty} m\ell^{-m}$$

with $C = vol(\{x \in O_F : |x| = 1\}) = 1 - \frac{1}{\ell}$, but

$$\sum_{m=0}^{\infty} m\ell^{-m} = \sum_{m=1}^{\infty} m\ell^{-m} = x\frac{d}{dx}\left(\sum_{m=0}^{\infty} x^m\right)\Bigg|_{x=\frac{1}{\ell}} = x\frac{d}{dx}\left(\frac{1}{1-x}\right)\Bigg|_{x=\frac{1}{\ell}} =$$

$$= \left(\frac{x}{(1-x)^2}\right)\Bigg|_{x=\frac{1}{\ell}} = \frac{1}{\ell(1-\frac{1}{\ell})^2} = \frac{\ell}{(\ell-1)^2}$$

To sum up,

$$\int_{O_F} val(x)dx = \frac{\ell}{(\ell-1)^2}\left(1 - \frac{1}{\ell}\right) = \frac{1}{\ell-1} = \frac{\deg(f)}{\ell-1}$$

Case 2: $f(x) = (x - \lambda)$ for $\lambda \in O_F$
Follows from previous case.

Case 3: $f(x) = (x - \lambda)$ for $\lambda \in O_{\bar{F}}$
Let $\lambda_0 \in O_F$ be s.t.

$$|\lambda - \lambda_0| = \min_{\mu \in O_F} |\lambda - \mu|.$$

It is easy to see that $val(x - \lambda) \leq val(x - \lambda_0)$ for any $x \in O_F$ and the assertion follows from previous case.

Case 4: $f(x) = (x - \lambda)^k$ for $\lambda \in O_{\bar{F}}$
Follows from previous case.

Case 5: General case
Follows from previous case and Lemma 13.4.7.

$\square$

Lemma 13.4.9. *The addition map $add : \mathbf{N} \times \underline{\mathfrak{z}} \to \underline{\mathfrak{g}}$ is finite.*

Proof. This follows from the following Cartesian square:

$$\begin{array}{ccc} \mathbf{N} \times \underline{\mathfrak{z}} & \xrightarrow{add} & \underline{\mathfrak{g}} \\ \downarrow & \square & \downarrow p \\ \underline{\mathfrak{z}} & \xrightarrow{p_{\underline{\mathfrak{z}}}} & \underline{\mathfrak{c}} \end{array}$$

$\square$

Lemma 13.4.10. *Let $m : \mathbb{A}^2 \to \mathbb{A}^1$ be the multiplication map. We have*

$$m_*(\mu_0^{\mathbb{A}^2,1}) = r(z)\mu_0^{\mathbb{A}^1,1},$$

where

$$r(z) = \frac{\ell - 1}{\ell}(val(z) + 1).$$

Proof.

$$m_*(\mu_0^{\mathbb{A}^2,1})(t^rO_F \smallsetminus t^{r+1}O_F) = \mu_0^{\mathbb{A}^2,1}(m^{-1}((t^rO_F \smallsetminus t^{r+1}O_F))) =$$
$$\sum_{i=0}^{r} \mu_0^{\mathbb{A}^2,1}((t^iO_F \smallsetminus t^{i+1}O_F)) \times (t^{r-i}O_F \smallsetminus t^{r-i+1}O_F)) =$$
$$= \sum_{i=0}^{r} \ell^{-r}\mu_0^{\mathbb{A}^1,1}(O_F \smallsetminus tO_F)^2 = \frac{(\ell-1)^2}{\ell^2}(r+1)\ell^{-r}$$

Using the action of $O_F^\times$, this implies the assertion. □

Proof of Theorem E. Let $\mathbf{N}^s \subset \underline{\mathfrak{g}}$ be the cone of non-regular nilpotent elements. Enumerate the nilpotent orbits $\{0\} = \mathbf{O}_1, \dots \mathbf{O}_m$ s.t. $\dim \mathbf{O}_i \leq \dim \mathbf{O}_j$ for any $i < j$. Let $\mathbf{N}_i = \bigcup_{j=1}^{i} \mathbf{O}_j$. Note that $\mathbf{N}_i$ is closed and $\mathbf{N}_0 = \emptyset$. Therefore, by Lemma 13.4.9, $\mathbf{N}_i + \underline{\mathfrak{z}}$ are closed (and $\mathbf{N}_0 + \underline{\mathfrak{z}} = \emptyset$). We will prove by down going induction on i that for any $i \geq 0$ the map $p|_{\underline{\mathfrak{g}} \smallsetminus (\mathbf{N}_i + \underline{\mathfrak{z}})}$ is an-FRS.

The base of the induction $i = m$ follows from Corollary 13.4.2. For the induction step, we assume the statement holds for $\mathbf{N}_{i+1}$ and prove it for $\mathbf{N}_i$. Let $\mathbf{U} = \underline{\mathfrak{g}} \smallsetminus (\mathbf{N}_{i+1} + \underline{\mathfrak{z}})$. Choose a nilpotent element $x \in \mathbf{O}_{i+1}(\mathbb{F}_\ell)$. Let $\mathbf{M}$ be the linear space given by Lemma 13.3.1 when applied to x.

Step 1. Reduction to $\mathbf{G} \times \mathbf{M}$.

Consider the map

$$\delta : (\mathbf{G} \times \mathbf{M}) \sqcup \mathbf{U} \to \underline{\mathfrak{g}} \smallsetminus (\mathbf{N}_i + \underline{\mathfrak{z}})$$

given on $\mathbf{G} \times \mathbf{M}$ by $\delta(g, l) := g \cdot (x + l)$ and on $\mathbf{U}$ by the embedding $\mathbf{U} \subset \underline{\mathfrak{g}} \smallsetminus (\mathbf{N}_i + \underline{\mathfrak{z}})$. By Lemma 13.3.1(1) it is submersive. Also, it is onto on the level of points over any field. Indeed, for any extension $E/\mathbb{F}_\ell$ we have $(\mathbf{O}_{i+1} + \underline{\mathfrak{z}})(E) = \mathbf{O}_{i+1}(E) + \underline{\mathfrak{z}}(E) = \mathbf{G}(E) \cdot (x + \underline{\mathfrak{z}})$ where the first equality is by Lemma 13.4.5. Thus,

$$(\underline{\mathfrak{g}} \smallsetminus (\mathbf{N}_i + \underline{\mathfrak{z}}))(E) = (\mathbf{O}_{i+1} + \underline{\mathfrak{z}})(E) \cup \mathbf{U}(E) =$$
$$= (\mathbf{G}(E) \cdot (x + \underline{\mathfrak{z}})) \cup \mathbf{U}(E) \subset \delta(((\mathbf{G} \times \mathbf{M}) \sqcup \mathbf{U})(E)).$$

Thus, by Corollary 13.4.6, it is enough to show that $p \circ \delta$ is an-FRS. Let $\delta' := \delta|_{\mathbf{G} \times \mathbf{M}}$. Notice that $p|_{\mathbf{U}}$ is an-FRS by the induction hypothesis. Therefore it is enough to show that $p \circ \delta'$ is an-FRS.

Notice that $\delta'(\mathbf{G} \times (\mathbf{M} \smallsetminus \underline{\mathfrak{z}})) \subset \mathbf{U}$ by Lemma 13.3.1(2). So, by Corollary 13.4.6 we deduce that $p \circ \delta'|_{\mathbf{G} \times (\mathbf{M} \smallsetminus \underline{\mathfrak{z}})}$ is an-FRS.

Step 2. Reduction to $\mathbf{M}$.

We can factor the map $p \circ \delta'$ as $p|_{(x+\mathbf{M})} \circ sh_x \circ pr_{\mathbf{M}}$, where $sh_x : \mathbf{M} \to x + \mathbf{M}$ is the shift map, and $pr_{\mathbf{M}} : \mathbf{G} \times \mathbf{M} \to \mathbf{M}$ is the projection. So, by Corollary 13.4.6 it is enough to show that $p|_{x+\mathbf{M}} \circ sh_x : \mathbf{M} \to \mathfrak{c}$ is an-FRS.

Also, by Corollary 13.4.6 we deduce that $p|_{x+(\mathbf{M} \smallsetminus \underline{\mathfrak{z}})} \circ sh_x$ is an-FRS.

Step 3. Proof that $p|_{x+\mathbf{M}} \circ sh_x : \mathbf{M} \to \mathfrak{c}$ is an-FRS when x is not subregular.
The assertion follows now from Lemma 13.4.4 and the condition on $\mathbf{M}$ given by Lemma 13.3.1(4,5).

Step 4. Proof that $p|_{x+\mathbf{M}} \circ sh_x : \mathbf{M} \to \mathfrak{c}$ is an-FRS when x is subregular.
In this case Lemma 13.4.4 is not applicable. So, we provide a direct argument. Using the action of $\mathbb{G}_m$ on $\mathbf{M}$ it is enough to show that $p_*(\mu_0^{x+\mathbf{M},1})$ has bounded density.

Without loss of generality we assume x is in Jordan form of type $(n-1, 1)$. More precisely $x = J_{n-1}(0) \oplus J_1(0)$.

Note that, until now, we only use the fact that $\mathbf{M}$ satisfies the conditions of Lemma 13.3.1.

So we can choose any such $\mathbf{M}$. Choose $\mathbf{M}$ as in the proof of Lemma 13.3.1. Explicitly,

$$\mathbf{M} = \{\ c(f) + (\alpha - 1)e_{n-1,n} + z | z \in \underline{\mathfrak{z}}; f \in \underline{\mathfrak{c}}; \alpha \in \mathbb{A}^1\}.$$

Here $c(f)$ is the companion matrix corresponding to $f \in \underline{\mathfrak{c}}$ where the coefficients are located in the first row.

Notice that $\mathbf{M}$ is the space of matrices of the form

$$\begin{pmatrix} * & 1 & & & & & \\ * & z & 1 & & & & \\ * & 0 & z & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \\ * & 0 & \cdots & 0 & z & 1 & \\ * & 0 & \cdots & 0 & 0 & z & \alpha \\ * & 0 & \cdots & 0 & 0 & 0 & z \end{pmatrix}$$

Denote

$$\mathbf{M}_1 = \{\ c(f) + (\alpha - 1)e_{n-1,n} | f \in \underline{\mathfrak{c}}; \alpha \in \mathbb{A}^1\}.$$

It is easy to see that for any $c(f) + (\alpha - 1)e_{n-1,n} \in \mathbf{M}_1$ we have

$$p(c(f) + (\alpha - 1)e_{n-1,n}) = f - f(0) + \alpha f(0).$$

Let us write $\underline{\mathfrak{c}} = \mathbb{A}^{n-1} \oplus \mathbb{A}^1$ where $\mathbb{A}^{n-1}$ represents the leading $n-1$ coefficients of an element in $\underline{\mathfrak{c}}$ and $\mathbb{A}^1$ corresponds to the constant term. Now we have

$$p_*(\mu_0^{x+\mathbf{M}_1,1}) = \mu_0^{\mathbb{A}^{n-1},1} \boxtimes m_*(\mu_{\mathbb{A}^2}),$$

where $m : \mathbb{A}^2 \to \mathbb{A}^1$ is the multiplication map.

This together with Lemma 13.4.10 implies that

$$p_*(\mu_0^{x+\mathbf{M}_1,1}) = \Phi \mu_0^{\mathfrak{c},1},$$

where $\Phi \in L^1(\mathfrak{c})$ is given by

$$\Phi(g) = \frac{\ell - 1}{\ell}(1 + val(g(0))),$$

where $g \in \underline{\mathfrak{c}}$.

From this we deduce $p_*(\mu_0^{x+\mathbf{M},1}) = h\mu_0^{\mathfrak{c},1}$, where $h \in L^1(\mathfrak{c})$ is given by

$$h(g) = \int_{O_F} \Phi(sh_z(g))dz = \frac{\ell-1}{\ell}\int_{O_F}(val(g(z))+1)dz.$$

Here sh_z is the shift by z of a polynomial.

The assertion follows now from Lemma 13.4.8 as

$$h(g) = \frac{\ell-1}{\ell}\int_{O_F}(val(g(z))+1)dz \le \frac{1}{\ell}deg(g) + \frac{\ell-1}{\ell} \le \frac{n}{\ell}+1$$

is a bounded function.

□

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References

[AA16] Avraham Aizenbud and Nir Avni. Representation growth and rational singularities of the moduli space of local systems. *Invent. Math.*, 204(1):245–316, 2016.

[AGKSa] Avraham Aizenbud, Dmitry Gourevitch, David Kazhdan, and Eitan Sayag. Effective local differential topology of algebraic varieties over local fields of positive characteristics. Preprint available at https://www.wisdom.weizmann.ac.il/∼dimagur/.

[AGKSb] Avraham Aizenbud, Dmitry Gourevitch, David Kazhdan, and Eitan Sayag. On Harish-Chandra's integrability theorem in positive characteristic. Preprint available at https://www.wisdom.weizmann.ac.il/∼dimagur/.

[Bha12] Bhargav Bhatt. Derived splinters in positive characteristic. *Compos. Math.*, 148(6):1757–1786, 2012.

[BL96] Joseph Bernstein and Valery Lunts. A simple proof of Kostant's theorem that $U(\mathfrak{g})$ is free over its center. *Amer. J. Math.*, 118(5):979–987, 1996.

[CGH14] Raf Cluckers, Julia Gordon, and Immanuel Halupczok. Local integrability results in harmonic analysis on reductive groups in large positive characteristic. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(6):1163–1195, 2014.

[DG67] Jean Dieudonné and Alexander Grothendieck. Éléments de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.*, 4, 8, 11, 17, 20, 24, 28, 32, 1961–1967.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[GH25] Itay Glazer and Yotam I. Hendel. A lower bound on the analytic log-canonical threshold over local fields of positive characteristic, 2025.

[Har98] Nobuo Hara. A characterization of rational singularities in terms of injectivity of Frobenius maps. *Amer. J. Math.*, 120(5):981–996, 1998.

[HC70] Harish-Chandra. *Harmonic analysis on reductive p-adic groups*, volume Vol. 162 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970. Notes by G. van Dijk.

[HC99] Harish-Chandra. *Admissible invariant distributions on reductive p-adic groups*, volume 16 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999. With a preface and notes by Stephen DeBacker and Paul J. Sally, Jr.

[Hes76] Wim H. Hesselink. Cohomology and the resolution of the nilpotent variety. *Math. Ann.*, 223(3):249–252, 1976.

[Hin91] V. Hinich. On the singularities of nilpotent orbits. *Israel J. Math.*, 73(3):297–308, 1991.

[Hum78] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[Kot05] Robert E. Kottwitz. Harmonic analysis on reductive p-adic groups and Lie algebras. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 393–522. Amer. Math. Soc., Providence, RI, 2005.

[Kov00] Sándor J. Kovács. A characterization of rational singularities. *Duke Math. J.*, 102(2):187–191, 2000.

[Lem96] Bertrand Lemaire. Intégrabilité locale des caractères-distributions de $\mathrm{GL}_N(F)$ où F est un corps local non-archimédien de caractéristique quelconque. *Compositio Math.*, 100(1):41–75, 1996.

[Lem97] Bertrand Lemaire. Intégrales orbitales sur $\mathrm{GL}(N, F)$ où F est un corps local non archimédien. *Mém. Soc. Math. Fr. (N.S.)*, (70):iv+94, 1997.

[Lem21] Bertrand Lemaire. Intégrales orbitales sur $GL(N, \mathbb{F}_q((t)))$. *J. Inst. Math. Jussieu*, 20(2):423–515, 2021.

[LW54] Serge Lang and André Weil. Number of points of varieties in finite fields. *Amer. J. Math.*, 76:819–827, 1954.

[Mus01] Mircea Mustaţă. Jet schemes of locally complete intersection canonical singularities. *Invent. Math.*, 145(3):397–424, 2001. With an appendix by David Eisenbud and Edward Frenkel.

[MvdK92] V. B. Mehta and Wilberd van der Kallen. On a Grauert-Riemenschneider vanishing theorem for Frobenius split varieties in characteristic p. *Invent. Math.*, 108(1):11–13, 1992.

[Pre03] Alexander Premet. Nilpotent orbits in good characteristic and the Kempf-Rousseau theory. volume 260, pages 338–366. 2003. Special issue celebrating the 80th birthday of Robert Steinberg.

[Ray78] M. Raynaud. Contre-exemple au “vanishing theorem” en caractéristique $p > 0$. In *C. P. Ramanujam—a tribute*, volume 8 of *Tata Inst. Fundam. Res. Stud. Math.*, pages 273–278. Springer, Berlin-New York, 1978.

[Rei18] Andrew Reiser. Pushforwards of measures on real varieties under maps with rational singularities, 2018.

[Ser06] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition.

[Slo80] Peter Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Smi97] Karen E. Smith. F-rational rings have rational singularities. *Amer. J. Math.*, 119(1):159–180, 1997.

[SS70] T. A. Springer and R. Steinberg. Conjugacy classes. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, volume Vol. 131 of *Lecture Notes in Math.*, pages 167–266. Springer, Berlin-New York, 1970.

[Sta25] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2025.

[WZ15] Richard L. Wheeden and Antoni Zygmund. *Measure and integral*. Pure and Applied Mathematics (Boca Raton). CRC Press, Boca Raton, FL, second edition, 2015. An introduction to real analysis.

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