-adic-Lecture-11

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p-adic-Lecture-11

11. Representations of Galois groups and (ϕ, Γ)-modules

K -padic

$\bar{K} \supset K$ $\qquad G_K = Gal(\bar{K}/K)$

$K_\infty \subset \bar{K}$ $\qquad H = Gal(\bar{K}/K_\infty)$

$\Gamma = Gal(K_\infty/K)$

$$1 \to H \to G_K \to \Gamma \to 1$$

$$H \cong Gal(\bar{E}_K/E_K)$$

E - field of char. p

Informal claim: for a field E of char. p it is easy to describe category $Rep(G_E)$

K - a field.

$Gal(\bar{K}/K)$ $\qquad \bar{K}_1 \cong \bar{K}_2$

$Gal(\bar{K}_1/K) \to Gal(\bar{K}_2/K)$

$Rep(G_K, \mathbb{F})$

Informal claim. Let E be a field of char. p.

Then we can explicitly describe $Rep(G_E, \mathbb{F}_p)$

I will describe some category $\Phi(E)$ and for every alg closure $E \supset K$ I will construct $F_E : \Phi(E) \xrightarrow{\sim} \mathrm{Rep}(\mathrm{Gal}(\bar{E}/K), \bar{E})$ compatible

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Digression.

Suppose we have comm. algebras A, B and a morphism

$\nu : A \to B$ $\qquad \mathrm{Spec} B \to \mathrm{Spec} A$

ν_* $\quad \mathrm{Res} : \mathcal{M}(B) \to \mathcal{M}(A)$

ν^* $\quad \mathrm{Ext} : \mathcal{M}(A) \to \mathcal{M}(B)$

$\qquad M \to B \otimes_A M$

claim Ext is canonically left adjoint to Res.

M is A-module, N B-mod

$\mathrm{Hom}_B(\mathrm{Ext}(M), N) = \mathrm{Hom}_A(M, \mathrm{Res} N)$

Suppose we are given $\varphi : A \to A$

Def. φ module over A is a module M over A together with semilinear morph.

$\varphi_M : M \to M$

E.g. $\varphi_M(am) = \varphi(a)\varphi_M(m)$

$\varphi_M : M \to \mathrm{Res}\, M$

$\varphi_M' : \mathrm{Ext}\, M \to M$

Def. φ_M is called *étale* if φ_M' is an isomorphism

Theorem Let E be a field of char. $p > 0$.

$\varphi : E \to E$ Frobenius morphism

$\varphi(a) = a^p$

Consider category $\Phi(E)$ of étale φ-modules over E

Then this category is

universal category of repr of Galois group over $\mathbb{F}_p$.

Suppose that E is not perfect.

Then we have example of étale φ-module over E

$M = E$, $\varphi_M = \varphi$.

$$M \to \operatorname{Res} M$$

φ_M^{lin} is an isom.

Construction of equivalence

$$\Phi(M)_E \xrightarrow{\sim} \operatorname{Rep}(\operatorname{Gal}(E^s/E), \mathbb{F}_p)$$

construction of functors V and D.

We have chosen some sep. completed extension E^s.

On E^s we have two actions.

(i) action of $\operatorname{Gal}(E^s/E)$

(ii) Operator Frobenius $a \to a^p$

They commute

$$V(M) = (E^s \otimes_E M)^{\varphi = 1}$$

$$D(W) = (E^s \otimes_{\mathbb{F}_p} W)^{\operatorname{Gal}(E^s/E)}$$

Technical reasons

$$E \subset E^s \subset \overline{E}$$

$\overline{E}/E^s$ is completely insep.

$\operatorname{Aut}(E^s/E) = \operatorname{Aut}(\overline{E}/E)$

<u>Thm.</u> Functors D, V give mutually inverse equivalence of categories $\Phi(E) = \operatorname{Rep}(\operatorname{Gal}(E^s/E), \mathbb{F}_p)$

$\mathbb{1}$ - triv. repr. $W = \mathbb{F}_p$.

$V(M) = (E^{s})^{\varphi=1}$

$D(E) = (E^{s})^{Gal(E_s/E)} = E$

Let $\varphi: M \to M$ be a φ-module over E (φ is semilinear). When is it étale?

(i) Suppose E is perfect. Then φ is étale if φ is a bijection.

(ii) Suppose $E' \supset E$.

Then we can consider

$M' = Ext_{E}^{E'}(M) = E' \otimes_E M$

M' has natural extension a semilinear φ, φ': $M \to M'$

exercise. (φ, M) is étale iff (φ, M') is étale

Criterion. $\varphi: M \to M$ semilinear. Then φ is étale $\Leftrightarrow$

(i) For some perfect ext. $E' \supset E$, ext. of φ is bijective

(ii) For any perfect ext. of E ext. (E', M') is bijective

(iii) $\varphi(M)$ generates M as E-module.

(iv) Choose a basis e_i of M and $\varphi(e_i) = \sum a_{ij} e_j$.

Consider matrix $A = (a_{ij})$.

φ is étale $\Leftrightarrow A$ is invertible

If we consider another basis f_i, and B - matrix $GL(d, E)$ relating bases

$A_f = B A_e \varphi(B)^{-1}$

$Rep(Gal(E^s/E), \mathbb{Z}_p)$ - objects are f. generated

$\mathbb{Z}_p$-modules with contin. action of $Gal(\bar{E}/E)$

I have to make a choice

Let us fix a complete DVR ring $O_\mathcal{E}$ with p uniformizer and $O_\mathcal{E}/(p) = E$, $\varphi : O_\mathcal{E} \to O_\mathcal{E}$, that lifts φ_E

E perfect $O_\mathcal{E} = W(E)$

We are interested only in $E = k((u))$, k - perfect of char p

Fix this algebra $O_\mathcal{E}$

Then we can make construction as before

Lemma. Let F/E be any separ. finite extension.

Then we can canonically construct $O_F \supset O_\mathcal{E}$ with similar properties.

Hence we can define

$O_{\mathcal{E}^{s}} = \bigcup_{F \subset E^{s}} O_F$

$\overline{O_{\mathcal{E}^{s}}}$ - completion.

construction Let us use this $\overline{O_{\mathcal{E}^{s}}}$ as period domain

Proposition. $(\overline{O_{\mathcal{E}^{s}}})^{\varphi=1} = \mathbb{Z}_p$

$(\overline{O_{\mathcal{E}^{s}}})^{Gal(E^{s}/E)} = O_\mathcal{E}$

$V : \Phi(E, O_\mathcal{E})^{et} \to Rep(Gal(E^{s}/E), \mathbb{Z}_p)$

$D : Rep(G, \mathbb{Z}_p) \to \Phi(E, O_\mathcal{E})^{et}$

Theorem. V, D are equiv. of categories, mutually inverse, compat. with $\otimes$, unit, duality.

Example, $E = k((u))$, k-pe

$\mathcal{O}_{\mathcal{E}} := \underline{W(k)}((u)) =$

$= \{ \sum_{n=-\infty}^{\infty} a_n u^n \mid a_n \to 0$ whe
$n \to -\infty$

$\mathrm{Rep}\,(\mathrm{Gal}(E^{s}/E), \mathbb{Z}_p)$

$\mathrm{Rep}\,(\mathrm{Gal}(E^{s}/E$

π - repr. of Gal o
$\mathbb{Q}_p$ - vector

φ - modules over

M - f. dim. vector
$\hat{\mathcal{E}}$, together
linear action
etale, an
$\mathcal{M} \exists\ \varphi$ - equivalent
