p-adic-Lecture-12

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12. Lecture 12. More about (ϕ, Γ)-modules

K - p-adic field, $K \subset \bar{K}$

$G_K = Gal(\bar{K}/K)$. $\mathbb{C}_p = \widehat{\bar{K}}$

$Rep(G_K, \mathbb{Q}_p)$

$Rep(G_K, \mathbb{Z}_p)$, $Rep(G_K, \mathbb{F}_p)$

$Rep(G_K, \mathbb{C}_p)$

$V: Rep(G_K, \mathbb{Q}_p) \to Rep(G_K, \mathbb{C}_p)$

$V(W) = W \otimes \mathbb{C}_p$

Period ring B

$Rep(G, \mathbb{Q}_p) \to Rep(G, B)$ [illegible]

$K \subset \mathcal{L} \subset \mathbb{C}_p$

$\mathcal{L} = K_\infty = K(\mu_{p^\infty})$

$H = Gal(\bar{\mathbb{Q}}_p / \mathcal{L})$

$\Gamma = Gal(\mathcal{L}/K)$

$1 \to H \to G_K \to \Gamma \to 1$

$Rep(G_K, \mathbb{Q}_p) \approx$

$Rep(\Gamma, Rep(H, \mathbb{Q}_p))$

E - field of char. p.

$G_E = \mathrm{Gal}(E^{sep}/E)$

$\mathrm{Rep}(G_E, \mathbb{F}_p) \simeq \varphi$-modules over E^{sep}

$U: W \mapsto W \otimes E^{sep}$

D: V - φ-mod. over E^{sep}

. $V \mapsto V^{\varphi=1}$

Thm: This is equiv. of

$\mathrm{Rep}(G_E, \mathbb{F}_p) \simeq \mathcal{M}(\varphi)^{et}$ over E^{sep}.

General ideology

$1 \to H \to G_K \to \Gamma \to 1$

$H = \mathrm{Gal}(E)$

$\mathrm{Rep}(G_K, \mathbb{F}_p) = \mathrm{Rep}\,\Gamma\, (\varphi\text{-mod over } E^{sep})^{et}$

$\mathrm{Rep}(G_K, \mathbb{Q}_p)$

$\mathrm{Rep}(G_K, \mathbb{Z}_p)$

Goal. construct a p-adic algebra A with action of Γ and an operator φ s.t. $\mathrm{Res}\, A = E^{sep}$.

$A^{\varphi=1} = \mathbb{Z}_p$

$A = W(E^{sep})$

Problem E is not perfect.

I. Replace E by $F = \mathrm{Perf}(E)$ - perfectization of E

$E \subset \bar{E}$, $R(\bar{E})$ - insepab. part of $\bar{E}$.

..... E $\qquad$ $E \xrightarrow{\sim} F$

$Gal_F = Gal_E$

$Rep(Gal(\bar{F}), \mathbb{F}_p)$ – φ-mod over F

$Rep(G_F, \mathbb{Z}_p) = \varphi$-mod. over $A = W(F)$

$K \subsetneq \mathcal{L} = K_\infty \subset \bar{K}$

Then we can construct E s.t. $Gal_E \simeq G$

Lift E^{sep} to algebra $\mathbb{Z}_p$ – A

$Rep(G, \mathbb{Z}_p) = Rep(E, \varphi)$ over A.

$\mathcal{L} = K_\infty$

A $\qquad$ E, E^{sep}

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d shalis, π - Povet.

Examples K_∞,

$E = k'((\pi))$

π - uniformizer in some way

$E \subset \tilde{E}$

$\tilde{\pi} \in W(\tilde{E})$

A = Generated by $\tilde{\pi}$ and Teich. (E)

K - finite extension of $\mathbb{Q}_p$

$\mathcal{L}$ = max. abelian extension of K

$K \subsetneq \mathcal{L} \subset \bar{K}$

$k' = k(\mathcal{L})$ – perfect field, of char p

$A = \mathcal{O}_K \otimes_{W(k)} W(k')$

$W(k)$ is $\overset{k' \supset k}{\ldots}$ ...

1) On A we have canonical φ - comes from Frobenius on $\tilde{E}$

2) We have canonical action of $\Gamma = Gal(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$

Per $[\mathbb{Z}_p[\varepsilon], \mathbb{Z}_p] = \mathbb{C}[\varepsilon, \Gamma]$ over A

$\Gamma = Gal(\varphi(K))=$

$= Gal(K^{ab}/K) = (K^{un})$ completion

$Rep(G_K, \mathbb{Q}_p) =$

$= (\varphi, \Gamma)$-modules over, étale $\mathcal{A} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} A$

1) φ is bijective $\varphi: M \to M$ - φ-linear, Γ-linear, bijective

2) $\exists$ φ-invariant lattice in M.

$Gal(K) \supset W_K$ - Weil group.

$I_K \to Gal(K) \to Gal(k) = \hat{\mathbb{Z}}$

$1 \to I_K \to W_K \longrightarrow \mathbb{Z} \to 1$

(?) $Rep(W_K, \mathbb{Q}_p) =$ (φ, Γ)-modules over $\mathcal{A}$, étale φ is bijective

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We did constructions for $\mathbb{Q}_p$, but we can do it more generally (Scholze)

Def. a perfectoid field is a field of char. 0

with a valuation $v: L \to \mathbb{Q} \cup \infty$

s.t. (1) L is complete wrt v.

(2) v is not discrete

(3) $Fr: \mathcal{O}_L / p\mathcal{O}_L \supseteq$ is onto

Typical example is $L = K_\infty$

Tilting construction

Suppose L is perfect.

$\mathcal{O}_L$ $\qquad \mathcal{O}_L^b = \varprojlim \mathcal{O}_L \qquad \mathcal{O}_L \xrightarrow{p} \mathcal{O}_L$

$\mathcal{O}_L^b = \{ (x_0, \dots x_n \dots) \mid x_i \in L \}$

$x_n = x_{n+1}^p$

Claim. $\mathcal{O}_L^b$ has natural structure of field of char. p.

It is perfect.

$v(x_0, \dots) := v(x_0) := \frac{v(x_n)}{p^n}$

v-valuation on L^b,

L^b - complete w.r.t. v

L^b is v-complete field perfect. It has a v-topology

$W(L^b) = \{ x_0, x_1 \dots x_n \dots (x_i \in L^b) \}$

$\sum p^i \, \mathrm{Teich}(x_i)$.

Field with p-adic valuation v_p

weak topology

$W(L^b) = \prod L^b_i$

$W(L^b)$ is similar $(\mathbb{F}_p[[t]][\alpha])$

$$W(L^b) = \{ \sum x_i p^i : x_i \in L^b \}$$

similar to

$L^b[[u]]$

$L^b = k((t)) = k[[t]][t^{-1}]$

$W(L^b)$ is similar to

$k[[t]][t^{-1}][[u]]$

$k[[t, u]]$

: $u = p + v$