p-adic-Lecture-13

Monday, 18 January 2021 13:19

p-adic-Lecture-13

13. Lecture 13. Summary

K - p-adic field

$G_K = Gal(\bar{K}/K)$.

$Rep(G, \mathbb{Q}_p)$ - cont. repr.

Rings of periods

B-ring, domain, with action of G.

$Rep(G_K, B)$ - cont., B-linear

$B^G = K$.

$Rep(G, K) \longleftrightarrow Rep(G, B)$

V: $W \longmapsto (B \otimes W)^{G_K}$

D: $M \longmapsto$

Def. W is B-admiss. if $\dim_K V(W) = \dim W$.

B-adm. repr. form a nice categ. in $Rep(G, K)$

Period rings.

1) $C = \hat{C}_K$ - closure of $\bar{K}$

2) $B_{HT} = \bigoplus_{q \in \mathbb{Z}} C(q)$

3) B_{DR}, filtered ring, $gr B_{DR} = B_{HT}$

$B_{DR} \supset B_{cris} \supset B_{st}$

$Rep(G_K; \mathbb{Q}_p)$ - description by (φ, Γ)-modules.

$K = \mathbb{Q}_p$

Deshalit, Porat.

$L = K^{ab} = \mathbb{Q}_p(\mu_\infty)$.

$G_K = Gal(\bar{K}/K) \supset W_K$

$1 \to I_K \to G_K \to Gal(\bar{\mathbb{F}}/\mathbb{F}) \cong \hat{\mathbb{Z}}$

$1 \to I_K \to W_K \to \mathbb{Z} \to 1$

Ansatz. W_K is more correct version of Galois group than G_K.

$Rep(W_K, K)$ $Rep(W_K, \mathbb{Q}_p)$

$L = K^{ab}$

$W(L/K) = K^{\times}$. CFT.

Theorem. $\mathrm{Rep}(W(L/K), \mathbb{Q}_p) = (\varphi, \Gamma)$-modules over ring A.

---

construction of the ring A.

consider field $L^{\flat}$; tilt of L - field of char. p.

$L^{\flat} = \{x_0, x_1, \dots, x_n, \dots \in L \mid x_{n+1}^p = x_n\}$

$(x+y)_i := \lim_{n \to \infty} (x_{n+i} + y_{n+i})^{p^n}$

F = perfect field.

claim. $F \simeq$ perfection of $\mathbb{F}_p((\omega))$,

$\omega = (p, \sqrt[p]{p}, \sqrt[p^2]{p}, \dots)$

$A \overset{?}{=} W(F)$

$K < K_\infty = K(\mu_{p^\infty}) < L = K(\mu_\infty)$

$F = K(\mu_{p^\infty})^{\flat}$ - perfection of $\mathbb{F}_p((\omega))$ closure

$L^{\flat}$ - alg. closure = = separ. closure of F.

res. $K_\infty = \mathbb{F}_p$

res $L = \overline{\mathbb{F}}_p$

$L = F^{ua}$

$A = \mathcal{O}_L \otimes_{W(k)} W(F)$

$K = \mathbb{Q}_p < K_1 = \mathbb{Q}_p(\mu_p) < K_2 = \mathbb{Q}_p(\mu_{p^2})$

$A = \mathcal{O}_{K_1} \otimes_{\mathbb{Z}_p} W(F)$.

On A we have action of 1) φ - Frobenius on F

$K = \mathbb{Q}_p$ 2) $\Gamma = W(L/K) = \mathbb{Q}_p^{\times}$

$\mathrm{Rep}(W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)) = (\varphi, \Gamma)$-mod over A

$p \in \mathbb{Q}_p^{\times}$ $\quad \rho(\gamma)$

p acts on A as φ.

Let M be a (φ, Γ)-module. $\varphi(p)/\rho(p)$ is an larger auto of M.

Tate module of W.

it is given by action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $\mu_{p^\infty} \simeq \mathbb{Q}_p(\mathbb{Z}_p)$

$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{End}(\mathbb{Q}_p/\mathbb{Z}_p) \simeq$ $\simeq \mathbb{Z}_p$

$\chi: W(\mathbb{Q}_p) \to \mathbb{Z}_p^{*}$

$\downarrow$

$W(\mathbb{Q}_p)^{ab} = W(\mathbb{Q}_p)/[\ ] \simeq \mathbb{Q}_p^{*}$

$\mathbb{Q}_p^{*} \to \mathbb{Z}_p^{*}$

$x \to x^{-1} \cdot p^{v(x)}$

φ, Γ $\quad A$

$Q(0)$ $\quad Q(1)$

$Q(0) \Leftrightarrow M = A$

action of Γ is action on A

action of φ is standard

$Q(1) \Leftrightarrow M = A$

$\rho(\gamma) =$

$\psi: W(K) \to \mathbb{Q}^{*}$

$\rho_1(\gamma) = \rho(\gamma)\, \psi(\gamma)$

$\psi: W(\mathbb{Q}_p) \to \mathbb{Q}_p^{*} \to$

$\downarrow$ $\quad x \mapsto |x|^{-1}$

$\mathbb{Q}_p^{*}$

$\psi|_{I_K} = \chi|_{I_K}$

$\psi: W(\mathbb{Q}_p)^{ab} = \mathbb{Q}_p^{*}$

$\mathrm{Rep}(G_K, \mathbb{Q}_p)$

$\mathrm{Rep}(W_K, \mathbb{Q}_p)$ — (φ, Γ)-mod

$1 \to I_K \to G_K \to \hat{\mathbb{Z}} \to 1$

$\cup$

$1 \to I_K \to W_K \to \mathbb{Z}$

Different point of view on W_K

K - finite extension of $\mathbb{Q}_p$.

Let L be a finite Galois extension of K.

$\Gamma = \mathrm{Gal}\, L/K$.

$1 \to G_L \to G_K \to \Gamma \to 1$

$1 \to W_L \to W_K \to \Gamma \to 1$

$E_L = [W_L, W_L]$

$1 \to W_L/E_L \to W_K/E_L \to \Gamma \to 1$

$1 \to L^{*} \to W_K/E_L \to \Gamma \to 1$

$W_K = \varprojlim_L W_K/E_L$

$\mathrm{Rep}(W_K, \mathbb{Q}_p)$ - $\mathbb{Q}_p$-repr.

Question What is (φ, Γ) description of the category $\mathrm{Rep}(W_K)$?