p-adic-Lecture-14

Monday, 25 January 2021 13:55

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14. Lecture 14. Example $K = \mathbb{Q}_p$

$K = \mathbb{Q}_p$.

$\mathrm{Rep}(G_K, \mathbb{Q}_p)$.

$\mathrm{Rep}(W_K, \mathbb{Q}_p)$

F - field of char. p

ω^α, $\alpha \in \mathbb{Z}[1/p]$

$F' = \mathbb{F}_p[\omega^\alpha]$

valuation ring.

$v(\omega) = 1$

F - completion of F'

$\sum a_m w^m \quad a_m \in \mathbb{F}_p$

for any $M \in \mathbb{M}$

only finite # of

$a_m \neq 0$ below M.

F - perfect field of char. p.

claim. $\mathbb{Q}_p^*$ naturally acts on F

$q \in \mathbb{Q}_p^*$

w^q

$q \in \mathbb{Z}[\frac{1}{p}]$

closure of $\mathbb{Z}[\frac{1}{p}] = \mathbb{Q}_p$

F - perfect field.

$A = W(F)[1/p]$ - A is a field of char. 0.

$\mathbb{Q}_p^*$ acts on A.

consider category

$Rep(\mathbb{Q}_p^*, A)$

Thm. $Rep(\mathbb{Q}_p^*, A)$ is equivalent to $Rep(G_K, \mathbb{Q}_p)$

Fix $\bar{K} \supset K$

consider sequences

x_i, $i \in \mathbb{Z}$, $x_i \in \bar{K}$ s.t.

$x_i = (x_{i+1})^p$

$x_i = 1$ when $i \le 0$

T group of such sequences.

i) $T \simeq \mathbb{Q}_p$

$T^+ = \{t \mid x_0 = 1, x_1 \in \bar{K}, \dots\}$

$T^+ \simeq \mathbb{Z}_p$ not canonically.

$\mathbb{Q}_p^*$ acts on T

transitively on $T \setminus \{1\}$.

Fix some elem. $\omega \in T^+ - \{1\}$

$E \to$

$K_\infty = K(\mu_{p^\infty})$.

$K_\infty^\flat = x_0, x_1, \dots \in K_\infty$ s.t.

$(x_{i+1})^p = x_i$

$K^\flat$ field of char p.

$\mathbb{F}_p(\omega) \to K^\flat$

$E \to K^\flat$

$W(E) \to W(K^\flat)$

$A \to W(K^\flat)(\frac{1}{p})$

: $\mathbb{Q}_p^*$-equivariant.

$\mathrm{Rep}(\mathbb{Q}_p^*, A) \Leftrightarrow \mathrm{Rep}(\mathrm{Gal}(\bar{K}/K), \mathbb{Q}_p)$

$W \to [A \otimes_{\mathbb{Q}_p} W]^{\mathbb{Z}_p^*}$ -repr.

$\mathrm{Rep}(\mathrm{Gal}(K_\infty/K), W(K_\infty^\flat)) \simeq$
$\mathrm{Rep}(\mathrm{Gal}(\bar{K}/K), \mathcal{O}$

$\mathbb{Z}_p^* = \mathrm{Gal}(K_\infty/K)$

in

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1) We have constructed a category C

2) Given any $\bar{K} \supset K$ and a choice of generator $\omega \in T(\bar{K}) \cdot 1$ we have canonical equivalence

$C \to \mathrm{Rep}(\mathrm{Gal}(\bar{K}/K), \mathbb{Q}_p)$

This correspondence is $\mathbb{Q}_p^*$ equivariant.

Example.

Suppose we want to construct some abel. category.

Usual way is to construct an algebra B and set $C = \mathrm{Mod}(B)$.

Sometimes this

simple ...

...nique method does not quite work.

Namely, suppose we have ~~some~~ explicit construction of

1) some group D

2) Some D-torsor Y

3) Some sheaf of algebras B on Y. D-equivariant.

Then we can consider category $C = D$-equiv. sheaves of B-modules

If I choose a point $y \in A$ then $C \simeq \mathrm{Mod}(B_y)$

Let us take A = algebra of D-equivariant sections of sheaf of algebras B.

Then our category $C = \mathrm{Mod}(A)$.

Our situation.

We

Given an algebraic closure $\bar{k} \supset k$

... where $K = \bar{K}$ we
constructed $T - 1$-dim.
space on $\mathbb{Q}_p$.
For any point
$\omega \in T \setminus 0$ we construct
category $\mathcal{C}_\omega$ in
equivariant way.
$\omega \longrightarrow A_\omega = W_{\mathbb{F}_p}(\omega^{d})[\frac{1}{p}]$
$\mathcal{C}_\omega = \mathrm{Rep}(\mathbb{Q}_p^{*}, A_\omega)$
$A = \mathbb{Q}_p$ - ~~...~~ subset of A_ω

$C = \mathrm{Mod}\ A$

$C = \mathrm{Rep}(\mathrm{Gal}(\bar{K}/K), \mathbb{Q}_p)$

$K = \mathbb{Q}_p$

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suppose $K/\mathbb{Q}_p$ - finite
extension. For simplicity
assume that $K/\mathbb{Q}_p$ - Galois.

$$\mathrm{Rep}(\mathrm{Gal}(\bar{K}/K), \mathbb{Q}_p).$$
$$\shortparallel$$
$$H$$

$C = \mathrm{Rep}(W_{\mathbb{Q}_p}, \mathbb{Q}_p)$. -
- sym. tensor category.
K is an object in
category C.
Moreover K is an
algebra object
$K \otimes K \to K$
equiv. $\mathrm{Rep}(H, \mathbb{Q}_p) =$
Modules over
algebra K in category
...

$C_0 = \mathrm{Rep}(W_{\mathbb{Q}_p}, \mathbb{Q}_p)$.

$W_{\mathbb{Q}_p}^{ab} \simeq \mathbb{Q}_p^*$

Given extension $K \supset \mathbb{Q}_p$ we have same description of $\mathrm{Rep}(W_K, \mathbb{Q}_p)$.

Hence we have a description of ℓ-adic repr.

Hence we have description of abelinization W_K^{ab}.

By CFT we know that $W_K^{ab} = K^*$

$\mathrm{Rep}(\mathbb{Q}_p^*, \mathbb{C})$

1) choice of π

2) choice of $\omega \in T(\pi)$

$C_0 = \mathrm{Rep}(\mathbb{Q}_p^*, A_0 \overset{def}{=} \mathbb{Z}[\omega^{\pm 1}])$