Witt Vectors:

- R Comm. ring, $V \in \mathrm{Mod}_R^{dbl}$, $T : V \circlearrowleft$ $\quad R \to V \otimes V^\vee \xrightarrow{T \otimes \mathrm{Id}} V \otimes V^\vee \to R.$

$\rightsquigarrow \mathrm{tr}(T) \in R.$

$\mathrm{tr}(T+S) = \mathrm{tr}(T \oplus S) = \mathrm{tr}(\varphi) + \mathrm{tr}(\psi)$

$\mathrm{tr}(T \otimes S) = \mathrm{tr}(T) \cdot \mathrm{tr}(S)$

$\mathrm{tr} : K_0(\{(V,T)\}) \longrightarrow R.$ $\quad (V,T) - (W,S)$ $\quad \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$

- R Comm. ring, $V \in \mathrm{Mod}_R^{fr}$, $T : V \circlearrowleft$

$P(T,t) = \det(I_V - tT) \in R[t]$ char pol.

- $P(T,t) = 1 + O(t)$
- $P(T \oplus S, t) = P(T,t) \cdot P(S,t)$
- $P(T \otimes S, t)$ is Complicated, "$P(T,t) = \prod(1+x_i t)$", "$P(S,t) = \prod(1+y_i t)$" $\rightsquigarrow$ "$P(T \otimes S, t) = \prod_{i,j}(1 + x_i y_j t)$"

(literally true over $\bar{F}$).

Want a ring w. "universal" char. Pol.

$A \xrightarrow{f} R$, $B \xrightarrow{g} R$ $\quad A \times B \xrightarrow{f \times g} R \times R \xrightarrow{m} R$, $\{ab, a \in A, b \in B\}$

$A, B \xrightarrow{(f,g)} R$

$A + B = A \sqcup B \xrightarrow{f \sqcup g} R$

Def: $\mathbb{W}(R) = R[[t]]_1^\times$,

(i) $(f+g)(t) = f(t)\,g(t)$

(ii) $(f \cdot g)(t) =$ mess!

if $f(t) \in R[t]$ we think of it as $\prod_{i=1}^{n}(1 + x_i t)$

for a multiset $A = \{x_i\} \subseteq R$.

(i) $A + B = A \sqcup B$

(ii) $A \cdot B = \{ab : a \in A, b \in B\}$. $\quad$ only for $R = \bar{F}$, otherwise only "locally over $\mathrm{spec}(R)$".

Universal prop:

$K_0(\{V,T\})$ is a λ-ring w. ring map

$\mathrm{tr} : K_0 \longrightarrow R.$

Def: λ-ring: R ring $+$ $\lambda_m : R \to R$,

$f : A \to R$, $\sum_{a \in A} f(a)$, $\quad \mathbb{W}(R) \xrightarrow{\mathrm{tr}} R$

- $\lambda_m(x+y) = \sum_{i+j=m} \lambda_i(x)\lambda_j(y)$
- $\lambda_m(xy) = \ldots$
- $\lambda_m \lambda_n(x) = \ldots$ $\Big\} \to$ mess. $\Big)$ $\quad \Lambda^n(V \otimes W)$, $\Lambda^m(\Lambda^n V)$

$R \longrightarrow \mathbb{W}(R)$

$(\sum_i a_i)(\sum_j b_j) = \sum_{i,j} a_i b_j$

$A \xrightarrow{f} R$

$\mathbb{W}(R)$ is a λ ring:

$\lambda_m(A) = \{a_1 \cdots a_m \mid a_i \neq a_j,\ a_i \in A\} = $

$$\begin{tikzcd} (A^m \setminus \Delta) \arrow[r] \arrow[d] & R^m \arrow[d] \\ (A^m \setminus \Delta)_{S_m} \arrow[r, "\lambda_m(f)"] & R \end{tikzcd}$$

Thm: $\mathrm{Forg} : \lambda\text{-Ring} \rightleftarrows \mathrm{Ring} : \mathbb{W}(-)$,

example: $\mathrm{Tr} : K_0(\{V,T\}) \longrightarrow R$, $\quad K_0$ is a λ-ring

$\Rightarrow P(-,t) : K_0(\{V,T\}) \longrightarrow \mathbb{W}(R).$

Witt Expansion & "cyclotomic circles"

for $a \in R$, $V_\ell(a) = \{b \in R : b^\ell = a\} = (1 + a t^\ell) \in R[[t]]$.

"cyclotomic orbit"

claim: $x \in \mathbb{W}(R) \Rightarrow x \stackrel{!}{=} \sum_{n \in \mathbb{N}} V_n(\omega_n)$

Proof: Want $f(t) = \prod_n (1 + \omega_n t^n)$, solve by induction on n.

Def: ω_n is the n-th Witt Components of $x = \sum_n V_n(\omega_n)$.

$\mathbb{W}(R) \ni x = a + \{\sqrt{b}, -\sqrt{b}\} + \{\sqrt[3]{c}, \zeta_3\sqrt[3]{c}, \zeta_3^2\sqrt[3]{c}\} + \{\sqrt[4]{d}, i\sqrt[4]{d}, -\sqrt[4]{d}, -i\sqrt[4]{d}\} + \ldots$

Adams Operations:

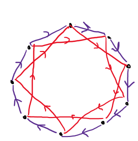
$A \subseteq R$, $\psi^\ell(A) = \{a^\ell \mid a \in A\}$ $\quad A \xrightarrow{f} R \xrightarrow{(\ )^\ell} R$, $\psi^\ell(f)$

claim: $\psi^\ell : \mathbb{W}(R) \longrightarrow \mathbb{W}(R)$ homomorphism.

(ghost Components: $g_m(A) = \mathrm{tr}(\psi^m(A)) = \sum_{a \in A} a^m$)

claim: $\psi^\ell(V_k(a)) = \gcd(k,\ell)\, V_{\frac{k}{\gcd(k,\ell)}}\left(a^{\frac{\ell}{\gcd(k,\ell)}}\right)$

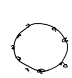
$$\boxed{\begin{aligned} \text{ex: } & (\ell,k) = 1 \Rightarrow \psi^\ell(V_k(a)) = V_k(a^\ell) \\ & \ell \mid k \Rightarrow \psi^\ell(V_k(a)) = \ell\, V_{\frac{k}{\ell}}(a) \\ & k \mid \ell \Rightarrow \psi^\ell V_k(a) = k\, V_1(a^{\frac{\ell}{k}}) \end{aligned}}$$



claim: $g_\ell(V_k(a)) = \begin{cases} 0, & k \nmid \ell \\ k a^{\frac{\ell}{k}}, & k \mid \ell \end{cases}$

Cor: If $R \in \mathrm{cAlg}_{\mathbb{Q}}$ then $\{g_\ell\}_{\ell \in \mathbb{N}}$ $\mathbb{W}(R) \xrightarrow{\sim} \prod_{\ell \in \mathbb{N}} R$ $\quad S_i = \sum x_i^\ell$

Analogy: $\mathrm{ch} : K(X) \otimes \mathbb{Q} \xrightarrow{\sim} \prod_i H^{2i}(X, \mathbb{Q})$ $\quad \sigma_j(\vec{x}) = \sum_I x_{i_1} \cdots x_{i_j}$



p-local case:

$\mathbb{W}(R) \simeq \prod_{(n,p)=1} \mathbb{W}_p(R) = \{\sum_{k=0}^{\infty} V_{p^k}(a)\},$

Proof: $x \longmapsto \{\sum_k V_{p^k}(\psi^{\bar{n}}(x))\}_{(n,p)=1}.$

example: $\vec{\omega}_{p^\infty}(x) = (x_0, x_1, \ldots)$, $\vec{\omega}_{p^\infty}(y) = (y_0, y_1, \ldots)$ $\Big\} \to \omega_{p^\infty}(x+y) = (x_0 + y_0,\ x_1 + y_1 + \frac{(x_1+y_1)^p - x_1^p - y_1^p}{p}, \ldots)$

Perfect field case: $x = \sum_k V_{p^k}(\omega_{p^k})$, $\quad \omega_{p^k} = (\omega'_{p^k})^{p^k} \Rightarrow$

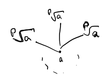
$\Rightarrow V_{p^k}(\omega_{p^k}) = p^k$ $\{\omega'_{p^k}, \omega'_{p^k}, \omega'_{p^k}, \ldots\}$ $= p^k[\omega'_{p^k}]$

$\Rightarrow x = \sum p^k[\omega'_{p^k}]$, Teichmuller Expansion. $\ldots(\cap)$

$\mathbb{W}(K)$ $\circlearrowleft$ Φ, V

<u>Frobenius</u> : $\psi^{\ell}(A) = \{a^{\ell} : a \in A\}$, $\quad \Phi(V_{\ell}(a)) = V_{\ell}(a^{\ell})$

<u>Verschiebung</u> (<u>Shift</u>) : $V_{\ell}(V_{\ell}(a)) = V_{\ell\ell}(a)$.



<u>Universal Property</u>:

<u>Def</u>: A <u>δ-ring</u>: is a ring R, +

- $\delta : R \to R$
- $\delta(0) = \delta(1) = 0$
- $\boxed{x^p + p\,\delta(x)} = \Phi(x)$ is a homomorphism.

  δ "a path" $\quad \Phi \bmod (p) \rightsquigarrow$ Frob.
- $\delta(x+y) - \delta(x) - \delta(y) = \dfrac{x^p + y^p - (x+y)^p}{p}$
- $\delta(xy) = p\,\delta(x)\delta(y) + x^p\delta(y) + y^p\delta(x)$.

$$A \xrightarrow{f} R$$

$$\big((A)^p - \Delta\big)/C_p \xrightarrow{\delta(f)} R$$

Σ_p

$\mathbb{W}_p(R)$ has δ-structure.

$$A \xrightarrow{f} R$$

$$\begin{tikzcd} \Delta A \arrow[r, hook] \arrow[rr, bend right, "\psi^p(f)"'] & A^p \arrow[r, "f"] & R \end{tikzcd}$$

$$\boxed{\delta(A) = \frac{A^p - \psi^p(A)}{p}}$$

<u>Thm</u>: Forg : δ-Ring$_{(p)}$ $\rightleftarrows$ Ring$_{(p)}$: $\mathbb{W}_p(-)$

$\mathbb{W}(R)$ $\qquad \delta_p(x) = \dfrac{x^p - \psi^p(x)}{p}$

$\delta_p \mapsto \psi^p$ is a δ_p map $\mathbb{W}(R) \to \mathbb{W}(R)$

$\psi^p \qquad \psi^p$