p-adic-6

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Algebra $B_{HT} = \sum_{q \in \mathbb{Z}} \mathbf{I}_B(q)$, where $B = C_K$.

Claim. *Hodge-Tate modules are modules admissible for the algebra B_{HT}.*

6. Lecture 6. Period ring B_{DR}.

7. Lecture 7. Witt vectors and the construction of the period ring B_{DR}.

7.1. Witt rings. We fixed a prime number p.

7.1.1. *Witt construction for p-algebras.*

Definition. 1. A p-ring is a discrete valuation ring B with a decreasing system of ideals b_i such that $b_i b_j \subset b_{i+j}$ that satisfies the following conditions

(i) $v(p) > 0$

(ii) B is complete with respect to ideals b_i and multiplication by p on B is injective.

(iii) The residue algebra $A = B/\mathfrak{p}$ is a perfect algebra of characteristic p.

2. We say that B is a strict p-ring if $b_i = p^i B$ for all $i \geq 0$

Claim. *The functor $B \mapsto A = B/pB$ defines an equivalence of categories between the category of strict p-rings and perfect algebras of characteristic p.*

The inverse functor $W : A \mapsto W(A)$ is called Witt construction.

7.1.2. *General Witt constriction.* Witt discovered that one can extend the functor W to all algebras. Namely to every algebra A he constructed in a functorial way an algebra $W(A)$ and a projection $pr : W(A) \to A$.

Let me describe his construction.

Given an algebra A consider the sets $S(A)$ and $T(A)$ - each isomorphic to the product $\prod_{i \in \mathbb{Z}_+} A_i$ of infinite number of copies of $A_i = A$. We denote by $s_i : S(A) \to A$ and $t_i : T(A) \to A$ the coordinate functions.

We define the Witt map of sets $W : S(A) \to T(A)$ by sequence of Witt polynomials w_i, namely $t_n = w_n(s_i)$ where

$$w_n = \sum p^i s_i^{p^{n-i}} \qquad \sum p^i s_i^{p^{n-i}}$$

.

In other words,

$w_o = s_0, \ w_1 = s_0^p + ps_1, \ w_2 = s_0^{p^2} + ps_1^p + p^2 s_2, \ldots$

Let us define on the set $T(A)$ the structure of the algebra with coordinate-wise addition and multiplication. We would like to introduce on the set $S(A)$ the structure of an algebra in such a way that the map $W : S(A) \to T(A)$ is a morphism of algebras.

Theorem 7.2. *There exists a unique way to define structures of algebras on sets $S(A)$ for all algebras A such that*

(i) These structures are functorial in A

(ii) The map $W : S(A) \to T(A)$ is a morphism of algebras for every algebra A.

The algebra $S(A)$ obtained in this way is called **the Witt algebra** of A and is usually denoted by $W(A)$.

$\diamond$

$A = \mathbb{Z}[x_\alpha] \subset$ ~~$\mathbb{Q}_p \mathbb{Z}_p$~~ $\mathbb{Z}'[x_\alpha] = A'$

$\mathbb{Z}' =$ encl. of $\mathbb{Z}$ by p

$\mathbb{Z}' \subset \mathbb{Q}$

$A \subset A'$

$S(A) \subset S(A')$

A-perf $\to W(A) \to$ given

~~A~~ $-\mathbb{Z}_p$-alg simple $\to W(A)$.

It is easy to check the following

Proposition 7.2.1. *1. Consider the natural projection $pr : W(A) \to A$ to the zero's component and denote by I its kernel. Then I is an ideal of $W(A)$, $W(A)$ is complete with respect to the powers of this ideal and for every n the $W(A)$-module I^n/I^{n+1} is canonically isomorphic to A.*

2. Let A be a perfect $\mathbb{F}_p$ algebra, (B, b_1) a p-ring. Then

$$Hom(W(A), B) = Hom(A, B/b_1)$$

$\Gamma_K = Gal(\bar K | K)$.

7.3. Construction of the period field B_{DR}. We start with a p-adic ~~ring~~ field K, consider $\bar{K}$, $C = C_K$.

Let $O = O_K$ be the ring of integers in C, $\mathfrak{m}$ its maximal ideal. Then $O/\mathfrak{m} = \bar{k}$ is an algebraically closed field.

$O = O_{C_K}$

Consider $_{\mathbf{p}}$-algebra A_O/pO and let $A = R(A_0)$ be its perfectization.

Claim. *The natural morphism $A \to A_0-$ canonically lifts to the morphism $W(A) \to O$*

$A_0 := O/pO$

$A \to A_0$

$W(A) \to O = O_C$

A_0 - not perfect.

$A = R(A_0) \to A_0$

$R(A_0) = \{a_0, a_1, \dots a_n, \dots \mid a_i \in A_0, \ a_{n+1}^p = a_n$.

$A_0 = O/pO$ not perfect

It is quite a long proof, but the main step is given by the following key lemma.

Lemma 7.3.1. *Let $\mathfrak{a} \subset O$ be any ideal containing pO*

Lemma 7.3.1. *Let $\mathfrak{a} \subset O$ be any ideal containing pO and topologically equivalent to pO. Then $R(O/\mathfrak{a}) = R(O/pO) = A$.*

To prove this we define the set $R(O)$ in the same way as $R(A_0)$ and show that as a set $R(O) = R(A/\mathfrak{a})$ for any ideal $\mathfrak{a}$ topologically equivalent to pO.

$A := R(O/pO)$.

$A = R(O/\mathfrak{a})$

$R(O) = \{a_0, a_1, \dots a_n, \dots \in O /$

$a_{n+1}^p = a_n$.

Sublemma. $R(O) \to R(O/\mathfrak{a})$ is a bijection.

$a_0, a_1, \dots a_n \in O/\mathfrak{a}$

can be lifted uniquely to $b_0, b_1, \dots b_n$ in O

Take a_n lift to some b and take $b_0 = b^{p^n}$

a_n approx by c_n, take $b_0^{(n)} = c_n^{p^n}$.

$\downarrow$

b_0

$\theta : W(A) \to O$

$\varphi : A = R(A_0) = R(O) \to O$

$W(A) \to O$

$\sum a_i p^i \to \sum \varphi(a_i) p^i$.

$\theta : W(A) \to O$ epimorphism.

$L = \ker \theta$.

$0 \to L \to W(A) \to O$

B^+_{dR} : completion on $W(A)$ w.r.t. L.

If t gener. elem. of $\ker \theta$

$0 \to L \to W(A) \to O$

L is generated by 1 elem. t.

$W(A) = \{a_0, a_1, \dots, a_n \dots\}, a_i \in A$

$\{0, 1, \dots \}$

A - char. p.

$W(A) \ni p = p \cdot 1$

$1 = (1, 0, 0, \dots 0)$

B - any algebra.

$W(B)$ $\quad \tilde{p} = (p, 0, \dots 0)$

$p = p \cdot 1$

$t = \tilde{p} - p$

$W(A) \to W(O) \qquad A \to R(O)$

$R(O) = (a_0, a_1, \dots \mid a_i = a_{i+1}^p)$
$\tilde{p} = (p, p^{1/p}, \dots)$
$R(O)^{-} = R(A)$

$W(A) \ni \tilde{p}$, $W(A) \ni p$

claim $\quad \tau = \tilde{p} - p$ is generator of $L = \ker \theta$

$B_{DR}^{A} :=$ completion of $W(A)$ w.r.t. powers of L.

$\dots I^2 \to I \to B_{DR}^{+} \to 0$

$p^{p^{-k}} \in K_\infty = K(\mu_{p^\infty})$

$\Gamma_\infty \subset \Gamma = Gal(\bar{K}/K_\infty)$

$\Gamma/\Gamma_\infty = Gal(K_\infty/K)$

$\to Aut(\mu_{p^\infty}) = \mathbb{Z}_p^*$

$L - c_1$

$m = L/L^2$. $- O$-module.

$B_{DR} = Frac(B_{DR}^+) = B_1$

B_{DR}

$O \subset C$

$R(A) \to A_0 = O/pO$

$\theta : W(A) \to O \qquad L = \ker \theta$

$W(A)$ compl. w.r.t. L

$\widehat{W(A)}$

$B_{DR}^{A} = \overline{W(A)} \left[\frac{1}{p}\right]$

$1 \to J \to BR^+ \to C \to 0$

$J/J^2 = C(1)$,

over $\mathbb{Q}_p$

$J/J^2 \ni$ line invariant w.r.t. action of Γ

and Γ acts there as cyclotomic character.

- B_{DR}^+ is a discrete val. ring,

J - max. ideal

$B_{DR}^+/J = C$

$J/J^2 = C(1) \ni \mathbb{Q}_p$-line invariant Γ

$BDR =$ Fract-field of BD_R^+

1) B_{DR} is a field.

$B_{DR} \simeq \mathbb{C}[[t]][t^{-1}]$

Invariant structures.

B_{DR} is discrete field.

B_{DR}^{+} – ring of integers

J – max. ideal.

J^{k} – filtration F of B_{DR}^{+} and B_{DR}

$$gr_F B_{DR} = \oplus J^{k}/J^{k+1} \simeq \oplus \mathbb{C}(k).$$

$$\cong B_{HT}$$

X – smooth proj. variety over K

$H_{et}(X_{\bar{K}}, \mathbb{Q}_p)$

$$B_{DR} \otimes_{\mathbb{Q}_p} H_{et}(X, \mathbb{Q}_p)^{G} = H_{DR}(X)$$

Using canonical lifting $\sigma : A = R(A_0) \to R(O)$ we define the map $\theta : W(A) \to O$.

Using properties of Witt construction we can see that this is a morphism of rings. It is easy to see that it is onto.

Let us describe the kernel $L = Ker(\theta)$.

Choose a primitive p-th root ξ of 1 and consider element $z = (1, \xi, ... \in R(O)$. Set $u = z - 1$

Claim. *(i)* $v(u) = p/p - 1$

(ii) The module $\mathbb{Z}_p u$ *does not depend on choices.*

Here p *odd. For* $p = 2$ *similar.*