p-adic-lecture-8

Monday, 14 December 2020 14:12

p-adic-lecture-8

8. Lecture 8. Tate-Sen theorems.

I would like to come back and discuss proofs of some results about the field $C = C_K$.

8.1. Basic results about the field $C = C_K$.

Theorem 8.2. ***1.*** *Field C is algebraically closed.*

2. *Let $H \subset Gal(\bar{K}/K)$ be a closed subgroup and $L = \bar{K}^H$ the corresponding subfield. Then $C^H = \check{L}$ – the closure of the field L in the topology of C.*

8.3. Galois structure of the field K_∞ and its closure $\check{K}_\infty$. One of the lessons we have seen from the construction of the field B_{DR} was that a very important role is played by the field K_∞ and its closure $\check{K}_\infty$. We will discuss this in more detail.

Fix a p-adic field K. For every integer $m \geq 0$ consider the group $\mu_{p^m} \in \bar{K}$ of roots of 1 of degree p^m and denote by μ'_{p^m} the subset of primitive roots. We denote by μ_{p^∞} the union of the groups μ_{p^m} for all m.

We denote by K_m the field extension $K_m = K(\mu_{p^m}$ and by K_∞ the union of all these fields.

Proposition 8.3.1. ***1.*** *For every m the field extension K_m/K is Galois and its Galois group is naturally embedded into the group $T_m = Aut(\mu_{p^m}$*

2. *The Galois group $Gal(K_\infty/K)$ is a closed subgroup of the group $D = Aut(\mu_\infty) = \mathbb{Z}_p{}^*$.*

3. *If $K = \mathbb{Q}_p$ then embeddings in 1 and 2 are isomorphisms.*

$K_m = K$ (roots of $x^{p^m} - 1$)

$$\Phi_m = \frac{x^{p^m} - 1}{x^{p^{m-1}} - 1} = 1 + x^{p^{m-1}} + \dots + x^{(p-1)p^{m-1}}$$

Roots of Φ_m are μ'_{p^m}

Claim: If $K = \mathbb{Q}_p$ then Φ_m is irreducible.

$[K_m : K] = (p-1)p^{m-1}$

$Gal(K_m/K) = Aut(\mu_m)$.

Claim: Let ξ_m - prim. p^m root of 1

then $v(1-\xi_m) = \dfrac{v(p)}{(p-1)p^{m-1}}$

Proof. $\prod_{\xi_m\text{ - prim. roots}} (1-\xi_m) = \Phi_m(1) = p$

$v(1-\xi_m)$ the same for all prim. roots.

$\sum_{\xi} v(1-\xi_m) = v(p)$

Fix a number m and consider the field $K_m \subset K_\infty$. We denote by D_m the Galois group $D_m = Gal(K_\infty/K_m)$. Let T_m be the subspace of elements $x \in K_\infty$ such that $tr_{K_m}(x) = 0$.

Claim. *We have canonical decomposition $K_\infty = K_m \oplus T_m$. This decomposition is invariant with respect to the Galois group D_m.*

In fact, this decomposition extends to the closure $\check{K}_\infty =$

$K_m \oplus \check{T}$, and is invariant with respect to the action of the Galois group H_m.

Theorem 8.4. *Fix $m > 0$.*

(i) The Galois group D_m is isomorphic to the group $\mathbb{Z}_p$.

(ii) We have a canonical decomposition $\check{K}_\infty = K_m \oplus \check{T}$ invariant with respect to the action of the group D_m

(iii) Choose a topological generator $d \in D_m$. Consider the operator$I = d - 1$ on $\check{K}_\infty$. Then the operator I annihilates K_m and is invertible on $\check{T}$.

$x \in K_\infty$ choose $K_n \ni x$

$tr_{K_n \to K_m}(x)$

$K_\infty = K_m \oplus T$

$RTr : K_\infty \to K_m$

$RTr(T) = 0$

RTr is Galois invariant.

$\hat{K}_\infty$ - completion of K_∞

$\hat{K}_\infty = K_m \oplus \hat{T}$

$RTr : \hat{K}_\infty \to K_m$

contin. morphism

$$RTr(x) := \lim_{n \to \infty} \frac{1}{p^{n-m}} tr_{K_n \to K_m}(x)$$

$x \in K_m$

$tr_{K_n \to K_m}(x) = p^{n-m} \cdot x$

$RTr(x) = x$

$K = \mathbb{Q}_p$ $\quad Gal(K_\infty / K) = \mathbb{Z}_p^* \cong \mathbb{F}_p^* \times \mathbb{Z}_p$

Claim. Let K be a p-adic field. Then $\mathrm{Gal}(K_{ab}/K)$ is a finite group $\times \mathbb{Z}_p$

Let N be a field, $H \subset \mathrm{Aut}(N)$

$M = N^H$.

$\mathrm{Rep}_N(H)$

$\mathrm{ind} : \mathrm{Vect}(M) \to \mathrm{Rep}_N(H)$

$W \to N \otimes_M W$

$\mathrm{res} : \mathrm{Rep}_N(H) \to \mathrm{Vect}(M)$

$V \mapsto W = V^H$

$\mathrm{ind}\,\mathrm{res}(V) \to V$

We say that V has descent to M if this is an isom.

We work with inf. H. H acts on N continuously.

Remark. If V has descent unto some subgroup $H' \subset H$ of finite index, then it has descent unto H.

---

v - p-adic field

$\overline{K}$ - alg. closure

$C = G_K = \widehat{\overline{K}}$ - closure of $\overline{K}$.

assume $K = \mathbb{Q}_p$.

1. $K \subset \widehat{K}_\infty \subset C$

Let $G_K = Gal(\overline{K}/K)$

$H \subset G_K$ - subgr. corresp. to K_∞

$\Gamma = G_K/H = Gal(K_\infty/K)$

Step 1. Proposition. Any cont. repr. V of the group G_K has descent w.r.t. H.

V^H - $\widehat{K}_\infty$-vector space.

$Rep_C(G_K) \simeq Rep_{\widehat{K}_\infty}(\Gamma)$

Step 2. Descent from $\widehat{K}_\infty$ to K_∞

$Rep_{\widehat{K}_\infty}(\Gamma) \simeq Rep_{K_\infty}(\Gamma)$

Step 3. How to go from repr. of Γ over K_∞ to repr. of ~~Γ~~ over K_∞.

Galois cohomology.

N, group H that acts on N, $M = N^H$

Let V be (H, N)-module of dim. d.

choose a basis e of N.

Then for every $h \in H$ we can consider basis mel of V.

$h(e) = A(h)\, e$,

$A(h) \in GL(d, N)$

The choose a different basis f. Then $f = Be$,

$B \in GL(d, N)$.

$\tilde{A}_f(h) = B\, A_e(h)\, h(B)^{-1}$

$A_e : H \to GL(d, N)$

$(*)$ $\quad c(h_1 h_2) = c(h_1) \cdot h_1(c(h_2))$

Isom (Repr. of dim d) =

= cocycles / ~~a~~ bound. equiv. relation.

$H^1(H, GL(d, N))$.

$H^1_{cont}(H, GL(d, N))$

Step 1.

$H^1_{cont}(\Gamma, GL(d, \hat{K}_\infty)) \cong H^1_{cont}(G_K, GL(d, \mathbb{C}))$

Step 2.

$H^1_{cont}(\Gamma, GL(d, K_\infty)) \cong H^1_{cont}(\Gamma, \hat{K}_\infty)$

Step 3.

$H^1(\Gamma, GL(d, K_\infty))$

Let c be a cocycle in $H^1(\Gamma, GL(d, K_\infty))$

$c : \Gamma \to GL(d, K_\infty)$,

Γ compact

$c : \Gamma \to GL(d, K_m)$

for some m.

Restrict to $\Gamma' \subset \Gamma$ of finite index

$\Gamma' \cong \mathbb{Z}_p$.

Choose a topological generator $\gamma \in \Gamma'$

c is an ...

... is uniquely determined by $c(\gamma) \in GL(d, K_\infty)$.

$c(\gamma) \in GL(d, K_n)$ for some n.

Restrict everything to a small subgp. $\Gamma' < \Gamma$

$\Gamma' < \Gamma \cap Gal(K_\infty, K_n)$.

Then the action of Γ' on the field K_n is trivial.

Hence action of Γ' on $GL(d, K_n)$ is trivial.

Hence $c : \Gamma' \to GL(d, K_n)$ is a homomorphism, continuous.

$c : \Gamma' \to GL(d, K_n)$.

is

$\mathbb{Z}_p$

Take $\lim_{x \in \mathbb{Z}_p \to 0} \frac{\log c(x)}{\log(\chi(x))}$

$\Gamma = Gal(K_\infty/K)^{\times} = \mathbb{Z}_p^{*}$

$D_{Sen} \in Mat(d, K_n)$.

Started with repn. V of $(Gal(K), \mathbb{C})$ constructed an unit operator on $Mat(d, K_n)$ defined up to conjugation.