p-adic-Lecture-9

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9. Lecture 9. Descent in Tate-Sen theory

Let K be a p-adic field. Fix its algebraic closure $\bar{K}$ and denote by C its completion in norm topology.

Denote by $K_\infty \subset \bar{K}$ the field extension of K by all roots of 1 of degrees p^n, and denote by L the completion of K_∞ in C.

We denote $G = Gal(\bar{K}/K)$ and consider the closed subgroup $H = Gal(\bar{K}/K_\infty \subset G$. We denote by Γ the quotient group $\Gamma = Gal(K_\infty/K) = G/H$.

9.1. Generalities on descent. Consider, more generally, the situation where a topological group G acts on a topological field C. Let H be a closed normal subgroup of G, $\Gamma = G/H$, $K = C^G$, $L = C^H$.

Let $Rep(G, C)$ denote the category of finite dimensional C-vector spaces with continuous semi-linear action of the group G. Similarly consider the category $Rep(\Gamma, L)$.

We say that there is a **descent** from C to L if these categories are naturally equivalent.

In general case, consider the pair of adjoint functors
$R : Rep(G, C) \to Rep)\Gamma, L$ and $I : rep(|gam, L)_{\to} Rep(G, C)$
given by

$$R(V) = V^H, \quad I(W) = C \otimes_L W$$

We have canonical morphisms of functors $i : Id \to R{\cdot}I$ and $j : I \cdot R \to Id$.

Functor I clearly preserves the dimension. We will see that always $\dim(R(V)) \leq \dim(V)$.

Proposition 9.1.1. *The following conditions are equivalent*

(i) Functors I and R are mutually inverse eauivalences of categories

(ii) Functor R preserves dimensions.

Indeed, functor R is left exact. If it preserves the dimension then it is exact and conservative.

Consider the adjunction morphism $j : I \cdot R \to Id$. I claim it is an isomorphism.

Since the functor R is conservative it is enough to show that its composition with the functor R, i.e. morphism of functors $R \cdot I \cdot R \to R$ is an isomorphism.

However, we know that the composition $R \to R \cdot I \cdot R \to R$ is an identity morphism and looking at dimensions we see that the morphism $R{\cdot}I{\cdot}R \to R$ is an isomorphism.

We have seen that not always we have a descent. Here is some criterion for the descent.

Claim. *Suppose that for every d the cohomology group $H^1_{cont}(H, GL(d, C)$ is trivial. Then there is a descent from C to L.*

Now let us come back to situation when $H = Gal(\bar{K}/K_\infty)$ and show that in this case we have a descent. This result is due to Tate and Sen. It reduces the study of the category $Rep(G, C)$ to the study of much simpler category $Rep(\Gamma, L)$. It is much simpler since the groups Γ is almost isomorphic to $\mathbb{Z}_p$.

Remark on the proof of this result in the paper by Brion and Conrad.

Let us recall some things from cohomology theory.

9.1.2. *Cohomology.* 1. Discrete groups. $H^0(G, A), H^1(G, A)$.

If A is a commutative group we can also define $H^i(G, A)$. One of definitions to use cochain complex $0 \to C^0 \to C^1 \to ...$ where C^i is the groupd of functions fro G^i to A.

Theorem 9.2. *Let M/L be a finite Galois extension of fields with the Galois group $G = Gal(M/L)$. Then*

(i) $H^i(G, M^+) = 0$ for all $i > 0$.

(ii) $H^1(G, GL(d, M)) = 1$ (Hilbert 90)

9.2.1. *Continuous cohomology.* The same definition with continuous functions.

Let us discuss different levels of acyclicity. Suppose we have a complex $aA \to B \to C$ with morphisms d, d'. We say that it is acyclic at place B if it satisfies the following condition

Acyclicity 0. $Kerd' = Imd$.

Now suppose that our groups are equipped with metrics and differentials d, d' are continuous (i.e. bounded) morphisms. Then we can impose some stronger conditions

Acyclicity 1. There exists a constant $C > 0$ such that If $b \in B$ is a cycle then there exists an $a \in A$ such that $da = b$ and $||A|| \leq C||b||$.

In fact, it is better to consider slightly stronger condition

Acyclicity 2. There exists a constant $C > 0$ such that for any $b \in B$ we can find an element $a \in A$ such that $IIa|| \leq C||b||$ and $||b - da|| \leq C||d'b||$

$||a|| \leq d||b||$

$||a|| \leq C||b||$

$||b - da|| \leq C||d'b||$

Consider the situation as before. Let a profinite group H constinuously act on the field $C = C_K$ Let us set $L = C^H$.

Theorem 9.3. *Suppose that the complex defining the continuous cohomology is strongly acyclic ,i.e. it satisfies the condition Acyclicity 2 at C^1. Then $H^1(H, GL(d, C) = 1)$.*

cont

$H^1_{cont}(\mathfrak{g}) = 0$ $\quad \mathfrak{g} = \mathrm{Lie}\, GL(d, C) = Mat(d, C)$.

strongly acyclic

Let c be a cocycle in $H^1(H, GL(d, C))$ contin.

<u>step 1</u>. Enough to find an open sgr. $H_0 \subset H$ s.t. $c|_{H_0}$ is trivial.

c corresponds to some repr (ρ, V) of H.

$\rho|_{H_0}$ is trivial.

So I can descent to ~~the~~ field $M \supseteq C^{H_0}$ ~ finite extension of L.

For finite extensions Hilbert 90.

$c : H \to C$ continu.

Choose small neighborhood of 1 in $GL(d, C)$ and take

$H_0 = \rho^{-1}(U)$

Choose on C ideal of O_C

$\|x\| = p^{-v(x)}$, $x \in C$.

$O_C = \{x \mid \|x\| \le 1\}$.

consider ideal

$I \subset O_C$, $I = \{x \mid \|x\| \le \frac{1}{2}\}$

$Mat(d, I)$.

$U = 1 + Mat(d, I) \subset GL(d, C)$.

$H_0 = \rho^{-1}(U)$.

$c|_{H_0}$ is trivial in coho

compare $H^1(H_0, U)$ and $H^1(H_0, M)$,

$M = Mat(d, I)$.

$M \simeq U$ $\quad m \to 1+m$

$c : H_0 \to U$, $c' : H_0 \to M$.

$c' = da + \varepsilon$, ε small, $a \in U$

$c \approx \tilde{c}$, s.t. $\|\tilde{c}\| \le \frac{1}{2}\|c\|$

$\|a\| \le \text{const} \cdot \|c\|$

$c_0 = c, \quad c_1, \quad c_2, \ldots$

$c_i \simeq \varepsilon, \quad \|c_i\| \to 0$

approx. by $\tilde{\sigma} : H/H_0 \to GL(d, M)$,
c, $H \to GL(d, \mathbb{C})$. $\quad M = \mathbb{C}^{H_0}$

Let us see how to prove this stronger acyclicity condition 2 in our case when L is the closure of the field K_∞.

Given a cochain $c \in C^1(H, C)$ we can approximate it by a function c' that is locally constant on H and lies in $\bar{K}$.

Hence we can assume that there exists a subgroup H_0 of finite index in H that corresponds to a finite field extension M/L such that our cocycle reduces to a cocycle $c' \in C^1(H', M)$, where $H' = H/H_0$ is a finite group.

So we should just check that for all these finite groups we can choose the same constant C in condition Acyclioc-ity 2.

This would follow from the following theorem due to Tate.

Theorem 9.4. *For any finite extension M/L we have $tr(O_M)$ contains the maximal ideal $\mathfrak{m}$ of the ring O_L.*

We will prove this later.

Now let us recall ho to prove the acyclicity of the cohomology $H^1(Gal(M/L), M)$.

Reminder from cohomology theory. Let $(C^\cdot, d)$ be a complex of abelian groups.

Homotopy is on operator $D : C^\cdot \to C^\cdot$ of degree $-!$. Such homotopy induces an endomorphism ν_D of the complex $C^\cdot$ via $\nu_D = dD + Dd$.

Morphism ν_D induces zero morphism on cohomologies. Thus, if thia morpism is identity (or is invertible) this would guaranty acyclicity. In fact this is the standard way to prove acyclicity.

Let M/L be a finite Galois extension. Let us recall how to prove that $H^i(G, M) = 0$ for $i > 0$. Choose an element $m \in M$ such that $tr(m) = 1$. Such an element defines a homotopy $D = D_m$. On 1 cochains it is given by $\sum c_i g_i \mapsto \sum c_i g_i(m)$. It is clear that $\nu_D = Id$ that implies acyclicity.

Now come back to our situation. Let $L \subset C$ as before, $M \subset C$ a finite Galois extension of L.

By Tate theorem we can choose an element $m \in M$

such that $tr(m) = 1$ and $||m|| < 2$. Then the corresponding homotopy D_m has norm ≤ 2 and hence Acyclicity 2 condition holds with the constant $c = 2$.

$H^1_{cont}(H, GL(d, \mathbb{C})) = \{1\}$ for all d.

~~Dosraut~~ Rep $(G, \mathbb{C})$ = Rep $(\Gamma, \ldots)$.

Is not reduced to $\mathrm{Vect}(K)$. Vect(K) = Rep (...)

Obstruction is given by Sen operator D_{sen}.

Interesting to consider W-vector space over L and $D_{sen} : V \to V$.

$\Phi : (V, \Theta : V \to V) \to$ Rep (Γ, L)

$L \otimes V$ define ... of Γ.

Comparison between p-adic fields and field in char. p.

If ... valuation field k ... field, then K is similar to $k[[t]]$

Let F be a local field.

... field

in char p.

We want to study its finite field extensions.

$F \subset \overline{F}$, $\overline{F} \supset F_{sep}$.

$\overline{F}$ contains F_{ins} – maximal insep. ext.

$F_{ins} = \{x \in \overline{F} \mid x^{p^n} \in F$ for each $n\}$.

$F \longrightarrow K$
char p char 0

$F_{ins} \to K_\infty$

M/K_∞ – finite ext.

$tr: M \to K_\infty$ is onto

$tr: \mathcal{O}_M \to \mathcal{O}_{K_\infty}$ is almost onto

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