## 3. Lecture 3. $p$-adic representations of the Galois groups.

From now on we fix a prime number $p$.

Let us fix a $p$-adic field K that is a finite extension of the field $\mathbb{Q}_p$.

$\lozenge$

**3.0.1.** *Unramified and tamely unramified extensions.* If $L/K$ is a finite field extension we define $n(L/K) = deg(L/K)$, $f = deg(l/k)$, $e$-ramification index of $L/K$.

**Claim.** $n(L/K) = f(L/K) \cdot e(L/K)$

**Proposition 3.0.2.** *For a tower of finite extensions $K \subset L \subset M$ we have product formulas*

$$n(M/K) = n(M/L)n(L/K), f(M/K) = f(M/L)f(L/K)$$

$$e(M/K) = e(M/L)s(L/K)$$

Unramified and tamely ramified extensions.

Fix an algebraic closure $K \subset \bar{K}$. We consider the intermediate field extensions

$$k \subset K^{un} \subset K^{tr} \subset \bar{K}$$

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On the side of Galois groups we get subgroups of the Galois group $Gal(K) := Gal(\bar{K}/K)$

$$I_K = Gal(\bar{K}/K^{un}), Wild_K = Gal(\bar{K}/K^{tr})$$

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$Gal(K) \supset I_K \supset Wild_K$

(i) $Wild_K$ is pro p group

(ii) $Gal(K)/I_K \simeq \hat{\mathbb{Z}}$

(iii) $I_K/Wild_K \simeq \hat{\mathbb{Z}}^{\neq p} = \prod_{\ell \neq p} \mathbb{Z}_\ell$

**3.0.3.** *Representations of the Galos group $Gal(K)$ over $\mathbb{Q}_l$.* $\lozenge$

$(\rho, V)$ – f. dim. repr. of $Gal(K)$
over $\mathbb{Q}_l$

Passing to finite ext.
can assume that $\rho(Wild) = 1$.

**3.0.4.** *p-adic representations of the Galois group* $Gal(K)$.

**Definition.** $p$-adic representation $\rho$ of the Galois group $Gal(K)$ is a continuous linear representation $\rho : Gal(K) \to GL(V)$, where $V$ is a finite dimensional vector space over $\mathbb{Q}_p$.

Usually we will denote by $d$ the dimension of the vector space $V$.

◊ Representations and geometric representations.

$Gal(K)$ acts on $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$

$\ell \neq p$

**3.0.5.** *Fontaine's strategy for studying Galois representations.* Motivation from Grothendieck hypothesis.

Let $X$ be a proper smooth variety over the field $K$. There should be a functor relating $H_{et}(X, \mathbb{Q}_p)$ and $H_D R)(X)$.

As we have seen in case of $\mathbb{C}$ probably some periods should be involved.

**3.0.6.** *Ring of periods.* According to Fontaine a **ring of periods** $B$ is a topological $\mathbb{Q}_p$-algebra $B$ equipped with a continuous action of the group $Gal(K)$ and some additional structures, compatible with this action.

We denote by $R$ the ring $B^{Gal(K)}$. Fontaine assumes that it is a field.

Then Fontaine defines a functor $D = D_B$ from the category $Rep(Gal(K))$ of $p$-adic representations of the Galois group to the category of $R$-vector spaces with additional structure

$$D(V) := (B \otimes_{\mathbb{Q}_p} V)^{Gal(K)}$$

D(V) inherits structure from B

**3.0.7.** *$B$-admissible representations.* We will check that always $\dim D_B(V) \leq d = \dim(V)$.

We say that a representation $(\rho, V)$ is **admissible** with respects to the ring periods $B$ if we have equality of dimensions..

Thus any ring of periods defines some subcategory $Rep_{B-adm}(Gal)K)$ of representations of the Galois group. In case of representations of geometric origin these subcategories are related to some geometric properties of underlying varieties.

## 3.1. Basic examples.

**3.1.1.** $B = \bar{K}$. In this case $B$-admissible representations are just smooth representations.

**3.1.2.** $B = C_K$. In this case $B$-admissible representations are representations with smooth restriction to the inertia subgroup $I_K$.

$\Diamond$

**3.1.3.** $B = B_{DR}$. One of the main theorems of $p$-adic Hodge theory is that in this case the functor $D_B$ relates etale $p$-adic cohomology and DeRham cohomology.

### 3.2. Hilbert Theorem 90.

$k$ - any field

$L|K$ - finite Galois extension.

$\Gamma = Gal(L|K)$. $\Gamma$-acts on $L$

Define $(\Gamma, L)$-space, $V$ with action of $L$ and $\Gamma$ that are compat.

$\gamma(e v) = \gamma(e) \cdot \gamma(v)$.

H90 Theorem. category $Vect(K) \simeq cat(\Gamma, L)$-spaces.

$V$ - $K$-vector space $\to V \otimes_K L$,

$W$ - $\Gamma, L$ space $\to (L \otimes W)^\Gamma$

char $k = 0$ $\quad L = \bar{K}$ $\quad \Gamma = Gal(\bar{k}|k) = Aut(\bar{K}|K)$.

$Vect(K) \simeq (\Gamma, \bar{K})$ - vect. spaces.

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$B = \bar{K}$.

Then $B$-admis. repr. are repr. smooth.

$C_K$.

$\bar{K}$ - alg. closure of $k$.

$v : \bar{K} \to \mathbb{Q}$ - unique valuation

$v|K$ - usual valuation. $v(p) = 1$

$C_K$ = completion of $\bar{K}$ w.r.t. $v$.

$\|x\| = p^{-v(x)}$

$C_K$ - completion of $\bar{K}$ wrt $\| \ \|_v$

1) $\hat{\mathbb{C}}_K$ is algebraically closed.

$$(\hat{\mathbb{C}}_K)^{Gal(\bar{K})} = K$$

$$B = \mathbb{C}_K$$

$B$ - admissible $\Leftrightarrow$ action of $I_K$ is smooth.

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$$1 \to I_K \to Gal(\bar{K}/K) \xrightarrow{pr} \hat{\mathbb{Z}} \to 1$$

$$\mathbb{Z} = Fr \in Gal(\bar{k}/k) \qquad q = \#(k)$$

$$x \to x^q \qquad Fr/q$$

$$W_K = pr^{-1}(\mathbb{Z})$$

$$1 \to I_K \to W_K \to \mathbb{Z} \to 1$$

$\ell/k$ - finite extension

$k$ - finite field

$Gal(\ell/k)$ - group with dist. elem $Fr$.

$$\mathbb{Z} \to \hat{\mathbb{Z}}$$

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$W_K$ - [illegible] is better candidate of $Gal(K)$

cont. admiss. repn.

$=$ conti. repn. of $\mathrm{Gal}(\bar{K}/K)$ on $V$

that is smooth on $I_K$ $\Leftrightarrow$

$\Rightarrow$ smooth repn. of $W_K$.

Hilbert 90. $L/K$, $\Gamma = \mathrm{Gal}(L/K)$

$$H^1(\Gamma, GL(d, L)) = 1$$

If $L = \bar{K}$ $\quad \Gamma = \mathrm{Gal}(\bar{K}/K)$

$$H^1(\Gamma, GL(d, L)) = 1$$

cont. cond. chains.

$$u : \Gamma \to GL(d, L)).$$

$$H^1(W_K, GL(d, L)).$$