4. Lecture 4. Some rings of periods

4.1. p-adic fields. We fixed a prime number p. Consider a complete valued field K of characteristic 0 with valuation $v : K \to \mathbb{Q} \cap \infty$ such that $v(p) = 1$.

Let O_K denote the ring of integers in K, $\mathfrak{p}_K$ its maximal ideal and $k = O_K/\mathfrak{p}_K$-its residue field of characteristic p.

We are mostly interested in the case when K is a finite extension of $\mathbb{Q}_p$. However, sometimes it is convenient to extend the residue field k to its algebraic closure. So we adopt the following terminology.

Definition. A **p-adic field** is a field K of characteristic 0 with a discrete valuation $v : K \to \mathbb{Q} \cup \infty$ such that K is complete with respect to v and its residue field k is a perfect field of characteristic p.

For p-adic field K we choose a uniformizer $\pi = \pi_K$, i.e. any generator of $\mathfrak{p}_K$ as O_K-module

Examples. 1. Any finite extension K of $\mathbb{Q}_p$ is a p-adic field.

2. Let K be a p-adic field. Fix an algebraic closure of K and consider the maximal unramified extension $L = K^{un}$ of K. This field has a discrete valuation v and its residue field l is isomorphic to the algebraic closure of the residue field k of K.

However, the field L is not complete with respect to v, so it is not a p-adic field. The completion L' of L with respect to v is a p-adic field with residue field $l = \bar{k}$.

Exercise. Let L be an algebraic field extension of a p-adic field K. Then the valuation v uniquely extends to a valuation of the field L. If L/K is a finite extension then L is a p-adic field with this valuation.

Teichmuller representatives.

K - p-adic field.

$$\mathcal{O}_K \underset{s}{\overset{\pi}{\rightleftarrows}} k = \mathcal{O}_K/\mathfrak{p}_K.$$

For any $x \in k$ $s_1(x)$ $s_2(x)$

Claim(n) $s_1(x) - s_2(x) \in \mathfrak{p}^n$

claim 1 clear.

claim $n \Rightarrow$ claim $n+1$

$x = y^p$

$s_2(y)^p = (s_1(y)+\delta)^p$ $s_2(y) = s_1(y) + \delta$ $\delta \in \mathfrak{p}^n$

$s_2(x) = s_1(x)$ -

Define $\tau^n(x)$

$y^{p^n} = x$

$\tilde{y}$ - lift of y defined up to $\mathfrak{p}$

$t^n(x) := q^{p^n}$

Choose uniformiz. $\pi = \pi_K$

$s(x) = \lim t^n(x)$.

Proposition Every elem. $x \in K$ can be uniquely written

$$\sum_{n \in \mathbb{Z}} s(a_n) \pi^n$$

$a_n = 0$ for $n << 0$

Let Γ be a topol. group.
B is $\mathbb{Q}_p$-algebra
and B is Γ-module and continuous.

$Rep_B(\Gamma)$ - finite gener. B-mod with cont. action of Γ

$\gamma(bx) = \gamma(b)\gamma(x)$

(Γ, B)-modules.

4.1.1. *Teichmuller representatives.* ♠

Proposition 4.1.2. *Let K be a p-adic field, O_K its ring of integers and $p : O_K \to k$ the residue map. Then there exists unique multiplicative section $s : k \to O_K$.*

*Element $s(a)$ is called the **Teichmuller representative** of an element $a \in k$*

Exercise. ♠

Let $\pi \in \mathfrak{p}_K$ be a uniformizer. Then any element $x \in K$ can be uniquely written as

$$x = \sum_{i \in \mathbb{Z}} s(a_i)\pi^i$$

where $a_i \in k$ and $a_i = 0$ for $i \ll 0$

Claim. category has tensor product

~~Sympr~~

$M \otimes_B N$ - again (Γ, B)-module.

If B is a field, then

M ~~inverse~~ exists dual.

$M \otimes_B \widetilde{M} \to B$ is Γ equiv.

Fontaine. K-p-adic field, fix $\bar{K}$

let $\Gamma = \mathrm{Gal}(\bar{K}/K)$.

Ring of periods. B be a $\mathbb{Q}_p$-algebra with action of Γ

assume that $K = B^\Gamma$ is a p-adic field.

Then we have a functor

$$D_B : \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma) \to \mathrm{Vect}_K$$

$$M \to B \otimes_{\mathbb{Q}_p} M \to (B \otimes_{\mathbb{Q}_p} M)^\Gamma$$

Def. M is called B-admiss. D_B-admissible if

$$\dim_{\mathbb{Q}_p} M = \dim_K D_B(M)$$

claim. If B is a field then always $\dim D_B(M) \le \dim$

$$\dim D_B(M) \le \dim M$$

Examples.

1. $B = \bar{K}$

claim. M is B-admissible iff it is smooth.

$M = (\Gamma, \mathbb{Q}_p)$-module

$$B \otimes_{\mathbb{Q}_p} M = \bigoplus_{K} (K \otimes_{\mathbb{Q}_p} M)$$

Example. $Q \subset K \subset N$ Finite Galois ext.

$Q \subset K$ - subfield.

D: $\mathrm{Rep}_Q(\Gamma) \to \mathrm{Vect}_K$ $\quad \Gamma = \mathrm{Gal}(N/K)$

Proposition: any (Γ, Q)-module M is admissible.

$$M \to M' = K \otimes_Q M \to N \otimes_K M' \to$$
$$\to (N \otimes_K M')^{\Gamma}$$

can assume $Q = K$

M - (Γ, K)-module. Γ acts trivially on K

$$\dim_K (N \otimes M)^{\Gamma} = \dim_K M$$

1) M - vect. space over K
2) has action of Γ
3) has action of Γ

choose a large field extension $N' = N \supset K$

(Γ, N)-module $\tilde{M} = N \otimes_K M$

consider (Γ, N')-module N'

Γ functions on Γ

$N' = L$

(Γ, L) -module $= F(\Gamma)$

Theorem $\mathrm{Rep}_\Gamma(L) \simeq \mathrm{Vect}_L$

Sheaves on Γ over L

$\mathrm{Mor}_{\mathrm{fields}/K}(N, L) = \Delta$ is Γ-torsor.

$L \otimes_K M = \bigoplus_{\delta \in \Delta} L \otimes_K M$

$n(\dots)$ = Δ [illegible]

Prop. Repr. $M \in \mathrm{Rep}_K(\Gamma)$ is adm

$\Gamma = \mathrm{Gal}(\bar{K}/K)$

iff M is smooth.

Reduce to finite stab.

$H \subset \Gamma$ s.gr. of finite index

s.t. H acts on M trivi

replace $\bar{K}$ by $\bar{K}^H = N$

$M \to (B \otimes M)^\Gamma = \mathrm{Hom}_\Gamma(M', B)$

$M' = M^*$

$B = \bar{K}$

If M' is not smooth

it has maximal

smooth quotient $Q(M)$

$0 \to K \to M' \to Q(M')$

$D(M) = \mathrm{Hom}_G(M', \bar{K})$

$\omega : M \to \bar{K}$

$\to Q(M)$

Period ring $C = C_K$ -

dim $D(M) \leq$ dim $\mathrm{Hom}(Q[M], \bar{K})$

$\leq$ dim $Q(M)$.

K- p-adic field.

$C = C_K$ = completion of $\bar{K}$.

$\Gamma = \mathrm{Gal}(\bar{K}|K)$ acts on $\bar{K}$

preserves valuation

$\Rightarrow$ acts on C

What are C-admissible

Theorem. 1. C is algebraically closed.

2. $C^\Gamma = K$

Moreover if H is a closed sgr. of Γ, L-corresp. field extension $K < L < \bar{K}$

C^H = closure of L

$B = C$ - period ring.

M $D_C(M)$ - K-modules

Theorem (Faltings).

Let X be smooth proj. over K

consid $X_{\bar{K}}$ $H = H_{et}(X_{\bar{K}}, \mathbb{Q}_p)$ -

M is repr. of $\Gamma = \mathrm{Gal}(\bar{K}/K)$.

$D_C(M) \cong \bigoplus C(-q) \otimes H^p_{DR}(X, \Omega^q)$ $p+q=n$

---

cyclotomic character

K p-adic field.

$\Gamma = Gal(\bar{K}|K)$

There exists canonical 1-dim. repr. of Γ over $\mathbb{Q}_p$

Lemma. $R_p = \mathbb{Q}_p/\mathbb{Z}_p$ - abel. group.

Then $Hom(R_p, R_p) \simeq \mathbb{Z}_p$

$P = G_m(K)$ $\qquad P(\bar{K}) = \bar{K}^*$

Def. $T_p(G_m(K)) = Mor(R_p, G_m(\bar{K})) = \lim$

Tate module $(\Gamma, \mathbb{Z}_p)$-module.

$T(G_m) = \varprojlim_n \mu_{p^n}(\bar{K})$

$$T = H^2_{et}(\mathbb{P}^1(K), \mathbb{Z}_p)$$

T_ℓ mod $= Hom(\mathbb{Q}_\ell/\mathbb{Z}_\ell, \mathbb{P}(\bar{K}))$.

$Rep_{\mathbb{Q}_\ell}(\Gamma)$.

$T_p = (\mathbb{Q}, L_p)$ $\qquad \chi_n = \chi^n$

$T_p^n = T_p \otimes \cdots \otimes T_p$ $\quad n > 0$

$T_p^n = (T_p^{-n})^*$ if $n \leq 0$

$T_p(G_a) > Mor(R_p = \mathbb{Q}_p/\mathbb{Z}_p, \bar{K}^*)$.

$\Vert$

$Mor(R_p, \mu_{p^\infty})$

$\mu_{p^\infty} \simeq R_p$