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5. Lecture 5. Some linear and semi-linear algebra

5.1. Abstract semi-linear algebra, Period rings and admissible representations. It is always easier to solve purely algebraic problems that do not deal with topology. So consider the following abstract situation.

Setup Fix a field F and an group Γ. We denote by $Rep_F(\Gamma)$ the category that consists of pairs (ρ, V), where V is a finite dimensional vector space over F and ρ is an F linear action of Γ on V.

We do not consider any topology here.

$Rep_F(\Gamma)$ is a symmetric tensor category over the field F, i.e on this category we have the natural tensor product $\otimes = \otimes_F$. Here $\mathbf{I}_F = (Id, F)$ is a unital object and for any object (ρ, V) we can define the dual object (ρ^*, V^*).

It turns out that a good way to study this category and some of its interesting subcategories is to consider its semi-linear generalization.

Namely, suppose we have a commutative F-algebra B with 1 and an action r of the group Γ on the F-algebra V.

Consider the category $Rep_B(\Gamma)$ that consists of pairs (ρ, M). where M is a finitely generated B-module and ρ is a semi-linear action of Γ on M, i.e. it satisfies $\rho(g)(bm) = r(b)\rho(m)$.

This is also a tensor category. Moreover, we have a natural tensor functor $E_B : Rep_F(\Gamma) \to Rep_B(\Gamma)$ given by $E_B(V) = B \otimes_F V$. The object $\mathbf{I}_B = E_B(\mathbb{F})$ is the unital object in this category.

Let $K := B^\Gamma$ be the ring of Γ-invariants in B. In fact it is better to define the ring K in categorical terms, namely

$$K = End(\mathbf{I}_B)$$

.

Assumption 1. We assume that K is a field.

We define the functor $D_B : Rep_F(|Gam) \to Vect(K)$ by $D_B(V) = Hom(\mathbf{I}_B, E_B(V) = E_B(V)^\Gamma$.

Note, that we have a canonical morphism $\alpha_V : D_B(V) \otimes_K \mathbf{I}_B \to E_B(V)$

5.2. Case of a field B. .

First consider the case when the algebra B is a field. Later we will also consider some rings that are not fields but have good properties

Proposition 5.2.1. *Suppose B is a field. then for any V the morphism $\alpha_V : D_B(V) \times_K \mathbf{I}_B \to E_B(V)$ is injective.*

Definition. An object $V \in Rep_F(\Gamma)$ is called B-admissible (or simply admissible) if the morphism $\alpha_V : D_B \otimes_K \mathbf{I}_B \to E_B(V)$ is an isomorphism.

Theorem 5.3. *Fix a field B as before.*

Consider the category $Rep_F(\Gamma)$ of F-representations of the group Γ and distinguish inside it the full subcategory $Adm_F(\Gamma)$ of B-admissible modules. Then

(i) The category $Adm_F(\gamma)$ is an abelian subcategory closed with respect to subquotients (but not extensions)

(ii) This is a tensor subcategory, closed with respect to tensor product, symmetric and exterior powers and duality.

Proof of the proposition

(i) Abelian K-linear category.

Finite length.

Socle of M, notation $Soc(M)$

Claim. *(i) Socle(M) is isomorphic to a direct sum of* *simple objects.*

(ii) For any simple object L we have $Hom(L, Soc(M) = Hom(L, M)$

Now let us consider any K linear category and let I be a simple object such that $K = End(I)$ I claim that for any object M we the natural morphism $Hom(I, M) \otimes_K I \to M$ is an injection.

Enough to check this for the submodule $Soc(M)$. In this case this is obvious.

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Proof of the Theorem.

Admissibility condition for V can be written as

$$\dim(D_B(V)) = \dim(V)$$

(i) Subquotients

(ii) Let W be admissible and V be any representation. I claim that we have a natural isomorphism

$$D_B(V) \otimes_K D_B(W) \approx D_B(V \otimes W)$$

.

Indeed, we have an isomorphism
$D_W(B) \otimes \mathbf{I}_B \to W \otimes_B \mathbf{I}_B$.
Hence

$$E_B(V \otimes W) \approx V \otimes W \otimes \mathbf{I}_B \approx V \otimes (E_B(W)) \approx E_B(V) \otimes D_B(W)$$

this implies that

$$D_B(V) \otimes_K D_B(W) \approx D_B(V \otimes W)$$

.

(iii) This implies compatibility with symmetric and exterior powers.

(iv) If L is one dimensional admissible, then L^* is also admissible.

$V^{\otimes n} \longrightarrow D_B(V)^{\otimes n}$

$Sym^n(V) \longrightarrow Sym^n D_B(V)$

$\Lambda^n(V) \longrightarrow \Lambda^n D_B(V)$

(v) Let $\dim V = d$. Consider $L = \Lambda^d(V)$. Then L is one dimensional admissible G-module, and hence L^* is is one dimensional admissible G-module., and hence L^* is admissible.

Now we see that $W^* \approx L^* \otimes \Lambda^{d-1}(V)$ is also admissible.

5.4. Regular period rings. Now consider more general case. Let B be an F-algebra with an action of the groups Γ. Let us assume the following conditions

Assumption 1. B is a domain.

We denote by C the field of fractions of B.

Assumption 2. $K = B^{\Gamma}$ coincides with C^{Γ}. In particular, it is a field.

Assumption 3. Let L be any one dimensional representation of Γ. Then any nonzero morphism $\beta : \mathbf{I}_B \to E_B(L)$ is an isomorphism.

Definition. The algebra B satisfying conditions 1-3 is called **regular**

Theorem 5.5. *Suppose B is a regular algebra. Then all the statements from theorem above hold for the functor D_B*

5.6. Hodge-Tate representations. Let us come back to the field $F = \mathbb{Q}_p$.

Let $\mathcal{L} \in Rep_F(\Gamma)$ be the Tate module. For any $q \in \mathbb{Z}$ we introduce q-twisting by

$$V(q); = V \otimes \mathcal{L}^{\otimes q}$$

Consider the ring of periods $B := C_K$.

A representation $V\imath Rep_F(\Gamma)$ is called **Hodge-Tate** if $D_B(V)$ is isomorphic to a direct sum of modules of type $\mathbf{I}_B(q)$.

Proof of 5.5.

Injectivity.

$D_B(V) \otimes I_B \to E_B(V)$ is injective.

$$\begin{array}{ccc} D_B(V) \otimes I_B & \to & E_B(V) \\ \downarrow & & \downarrow \\ D_C(V) \otimes I_C & \to & E_C(V) \end{array}$$

We have seen that by Faltings' theorem all cohomology groups of bprojective smooth varieties are of his type. In fact it is possible to describe the category of Hodge-Tate modules as the category of admissible modules with respect to some algebra B_{HT}.

This is based on the following theorem.

Theorem 5.7. *$Hom(\mathbf{I}_B(q), \mathbf{I}_B(r) = 0$ if $q \neq r$ and eqials K if $q = r$*

$Ext^1(\mathbf{I}_B(q), \mathbf{I}_B(r)) = 0$ nif $q \neq r$ and is one dimensional over the field K if $q = r$

X/K - alg. variety, proj. smooth

$H^i_{et}(X_{\bar{K}})$ – Γ-module over $\mathbb{Q}_\ell$

$\Gamma = Gal(K)$

$C_K \otimes I_{C_K}(q)$

$B = C_K$

I_B $\mathcal{L}$-Tate module

$q \in \mathbb{Z}$ $I(q) = I \otimes \mathcal{L}^{\otimes q}$

$\Gamma = Gal(\overline{K}/K)$

$Hom(C_K, C_K(q)) = 0$ when $q \neq 0$

$C(q)^G = 0$ $q \neq 0$

$C_K^G = K$

Algebra $B_{HT} = \sum_{q\in\mathbb{Z}} \mathbf{I}_B(q)$, where $B = C_K$. $\simeq C_K[t, t^{-1}] = B_{HT}$

Claim. *Hodge-Tate modules are modules admissible for the algebra B_{HT}.*

B_{HT} is regular algebra.

Thm. (Tate)

Hom (K

$I(q)^{Gal} = 0$ $q \neq 0$

$= K$ $q = 0$

B_{DR}
