VANISHING OF CERTAIN EQUIVARIANT DISTRIBUTIONS ON p-ADIC SPHERICAL SPACES, AND NON-VANISHING OF SPHERICAL BESSEL FUNCTIONS

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Abstract. We prove vanishing of distribution on p-adic spherical spaces that are equivariant with respect to a generic character of the nilradical of a Borel subgroup and satisfy a certain condition on the wave-front set. We deduce from this non-vanishing of spherical Bessel functions for Galois symmetric pairs.

1. Introduction

Let $\mathbf{G}$ be a reductive group, quasi-split over a non-Archimedean local field F of characteristic zero. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$, and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Let $\mathbf{H}$ be a closed subgroup of $\mathbf{G}$. Let G, B, U, H denote the F-points of $\mathbf{G}, \mathbf{B}, \mathbf{U}, \mathbf{H}$ respectively. Suppose that $\mathbf{H}$ is an F-spherical subgroup of $\mathbf{G}$, i.e. that there are finitely many $B \times H$-double cosets in G. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G, H respectively. Let ψ be a non-degenerate character of U and let χ be a (locally constant) character of H. For $x \in G$ denote $H^x := xHx^{-1}$ and denote by χ^x the character of H^x defined by conjugation of χ. For a $B \times H$-double coset $\mathcal{O} \subset G$ define

$$\mathcal{O}_c := \{x \in \mathcal{O} \mid \psi|_{H^x \cap U} = \chi^x|_{H^x \cap U}\}.$$

Let

$$Z := \bigcup_{\mathcal{O} \text{ s.t. } \mathcal{O} \neq \mathcal{O}_c} \mathcal{O}.$$

Identify T^*G with $G \times \mathfrak{g}^*$ and let $\mathcal{N}_{\mathfrak{g}^*}$ be the set of nilpotent elements in $\mathfrak{g}^*$.

Consider the action of $U \times H$ on G given by $(u,h)x = uxh^{-1}$. This gives rise to an action of $U \times H$ on the space $\mathcal{S}(G)$ of Schwartz (i.e. locally constant compactly supported) functions on G and the dual action on the space of distributions $\mathcal{S}^*(G)$.

In this paper we prove the following theorem.

Theorem A (see Section 3)**.** *Let $\xi \in \mathcal{S}^*(G)^{(U \times H, \psi \times \chi)}$ be an equivariant distribution on G, i.e. $(u,h)\xi = \psi(u)\chi(h)\xi$. Suppose that the wave-front set (see section 2.2) $WF(\xi)$ lies in $G \times \mathcal{N}_{\mathfrak{g}^*}$ and* $\mathrm{Supp}(\xi) \subset Z$. *Then $\xi = 0$.*

In the case when $\mathbf{H}$ is a subgroup of Galois type we can prove a stronger statement. By a subgroup of Galois type we mean a subgroup $\mathbf{H} \subset \mathbf{G}$ such that

$$(\mathbf{G} \times_{\mathrm{Spec}F} \mathrm{Spec}E, \mathbf{H} \times_{\mathrm{Spec}F} \mathrm{Spec}E) \simeq (\mathbf{H} \times_{\mathrm{Spec}F} \mathbf{H} \times_{\mathrm{Spec}F} \mathrm{Spec}E, \Delta\mathbf{H} \times_{\mathrm{Spec}F} \mathrm{Spec}E)$$

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for some field extension E of F, where $\Delta\mathbf{H}$ is the diagonal copy of $\mathbf{H}$ in $\mathbf{H} \times_{\operatorname{Spec} F} \mathbf{H}$.

Corollary B (see Section 4). *Let $\mathbf{H} \subset \mathbf{G}$ be a subgroup of Galois type, and let χ be a character of H. Let S be the union of all non-open $B \times H$-double cosets in G. Let $\xi \in \mathcal{S}^*(G)^{(U\times H,\psi\times\chi)}$. Suppose that $WF(\xi) \subset G \times \mathcal{N}_{\mathfrak{g}^*}$ and* $\operatorname{Supp}(\xi) \subset S$. *Then $\xi = 0$.*

Note that if χ is trivial, we can consider the distribution ξ as a distribution on G/H. Considering $\tilde{\mathbf{G}} := \mathbf{G}\times\mathbf{G}$ and taking $\mathbf{H}$ to be the diagonal copy of $\mathbf{G}$ we obtain the following corollary for the group case.

Corollary C (see Section 4). *Let ψ_1 and ψ_2 be non-degenerate characters of U. Let $B\times B$ act on G by $(b_1, b_2)g := b_1 g b_2^{-1}$. Let S be the complement to the open $B\times B$-orbit in G. For any $x \in G$, identify T_xG with $\mathfrak{g}$ and T^*_xG with $\mathfrak{g}^*$. Let*

$$\xi \in \mathcal{S}^*(G)^{U\times U,\psi_1\times\psi_2}$$

and suppose that $WF(\xi) \subset S \times \mathcal{N}_{\mathfrak{g}^*}$. Then $\xi = 0$.

1.1. **Applications to non-vanishing of spherical Bessel functions.** Let π be an admissible representation of G (of finite length), and $\tilde{\pi}$ be the smooth contragredient representation. Let $\mathbf{H} \subset \mathbf{G}$ be an algebraic spherical subgroup and let χ be a character of H. Let $\phi \in (\pi^*)^{(U,\psi)}$ be a (U,ψ)-equivariant functional on π and v be an (H,χ)-equivariant functional on $\tilde{\pi}$. For any function $f \in \mathcal{S}(G)$, we have $\pi^*(f)\phi \in \tilde{\pi} \subset \pi^*$.

This enables us to define the *spherical Bessel distribution* corresponding to v and ϕ by

$$\xi_{v,\phi}(f) := \langle v, \pi^*(f)\phi\rangle.$$

By [AGS, Theorem A] we have $WF(\xi_{v,\phi}) \subset G \times \mathcal{N}_{\mathfrak{g}^*}$.

The *spherical Bessel function* is defined to be the restriction $j_{v,\phi} := \xi_{v,\phi}|_{G-S}$, where S is the union of all non-open $B \times H$-double cosets in G. One can easily deduce from [AGS, Theorem A] and Lemma 3.1 that $j_{v,\phi}$ is a smooth function. Theorem A and Corollary B imply the following corollary.

Corollary D. *Suppose that π is irreducible and v, ϕ are non-zero. Then*

(i) For any open subset $U \subset G$ that includes $G \setminus Z$ we have $\xi_{v,\phi}|_U \neq 0$.
(ii) If $\mathbf{H}$ is a subgroup of Galois type then $j_{v,\phi} \neq 0$.

For the group case this corollary was proven in [LM, Appendix A].

1.2. **Related results.** In [AG] a certain Archimedean analog of Theorem A is proven (see [AG, Theorem A]). This analog implies that the Archimedean analog of Corollary D(ii) holds for any spherical pair (G, H) (see [AG, Corollary B]).

Corollary C together with [AGS, Theorem A] can replace [GK75, Theorem 3] in the proof of uniqueness of Whittaker models [GK75, Theorem C].

Theorem A can be used in order to study the dimensions of the spaces of H-invariant functionals on irreducible generic representations of G (see [AG, §1.3] for more details). It can also be used in the study of analogs of Harish-Chandra's density theorem (see [AGS, §1.7] for more details).

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2. Preliminaries

2.1. **Conventions.**

- We fix $F, \mathbf{G}, \mathbf{B}, \mathbf{U}, \mathbf{X}$ and ψ as in the introduction.
- All the algebraic groups and algebraic varieties that we consider are defined over F. We will use capital bold letters, e.g. $\mathbf{G}, \mathbf{X}$ to denote algebraic groups and varieties defined over F, and their non-bold versions to denote the F-points of these varieties, considered as l-spaces or F-analytic manifolds (in the sense of [Ser64]).
- When we use a capital Latin letter to denote an F-analytic group or an algebraic group, we use the corresponding Gothic letter to denote its Lie algebra.
- We denote by G_x the stabilizer of x and by $\mathfrak{g}_x$ its Lie algebra.
- For an F-analytic manifold X, a submanifold $Y \subset X$ and a point $y \in Y$ we denote by $CN_Y^X \subset T^*X$ the conormal bundle to Y in X, and by $CN_{Y,y}^X$ the conormal space at y to Y in X.
- By a smooth measure on an F-analytic manifold we mean a measure which in a neighborhood of any point coincides (in some local coordinates centered at the origin) with some Haar measure on a closed ball centered at 0. A Schwartz measure is a compactly supported smooth measure.
- The space of generalized functions $\mathcal{G}(X)$ on an F-analytic manifold X is defined to be the dual of the space of Schwartz measures. One can identify $\mathcal{G}(X)$ with $\mathcal{S}^*(X)$ by choosing a smooth measure with full support.
- Let $\phi : X \to Y$ be a submersion of analytic manifolds. Note that the pushforward of a Schwartz measure with respect to ϕ is a Schwartz measure. By dualizing the pushforward map we define the pullback map $\phi^* : \mathcal{G}(Y) \to \mathcal{G}(X)$.

2.2. **Wave front set.** In this section we give an overview of the theory of the wave front set as developed by D. Heifetz [Hef85], following L. Hörmander (see [Hör90, §8]). For simplicity we ignore here the difference between distributions and generalized functions.

Definition 2.1.

(1) *Let V be a finite-dimensional vector space over F. Let $f \in C^\infty(V^*)$ and $w_0 \in V^*$. We say that f* vanishes asymptotically in the direction of w_0 *if there exists $\rho \in \mathcal{S}(V^*)$ with $\rho(w_0) \neq 0$ such that the function $\phi \in C^\infty(V^* \times F)$ defined by $\phi(w, \lambda) := f(\lambda w) \cdot \rho(w)$ is compactly supported.*

(2) *Let $U \subset V$ be an open set and $\xi \in \mathcal{S}^*(U)$. Let $x_0 \in U$ and $w_0 \in V^*$. We say that ξ is* smooth at (x_0, w_0) *if there exists a compactly supported non-negative function $\rho \in \mathcal{S}(V)$ with $\rho(x_0) \neq 0$ such that the Fourier transform $\mathcal{F}^*(\rho \cdot \xi)$ vanishes asymptotically in the direction of w_0.*

(3) *The complement in T^*U of the set of smooth pairs (x_0, w_0) of ξ is called the* wave front set of ξ *and denoted by $WF(\xi)$.*

(4) *For a point $x \in U$ we denote $WF_x(\xi) := WF(\xi) \cap T_x^*U$.*

Remark 2.2.

(1) Heifetz defined $WF_\Lambda(\xi)$ for any open subgroup Λ of $F^\times$ of finite index. Our definition above differs slightly from the definition in [Hef85]. They relate by

$$WF(\xi) - (U \times \{0\}) = WF_{F^\times}(\xi).$$

(2) Though the notion of Fourier transform depends on a choice of a non-degenerate additive character of F, this dependence effects the Fourier transform only by dilation, and thus does not change our notion of wave front set.

Proposition 2.3 (see [Hör90, Theorem 8.2.4] and [Hef85, Theorem 2.8])**.** *Let $U \subset F^m$ and $V \subset F^n$ be open subsets, and suppose that $\phi : U \to V$ is an analytic submersion. Then for any $\xi \in \mathcal{S}^*(V)$, we have*

$$WF(\phi^*(\xi)) \subset \phi^*(WF(\xi)) := \left\{(x,v) \in T^*U | \, \exists w \in WF_{\phi(x)}(\xi) \text{ s.t. } d^*_{\phi(x)}\phi(w) = v\right\}.$$

Corollary 2.4. *Under the assumption of Proposition 2.3 we have*

$$WF(\phi^*(\xi)) = \phi^*(WF(\xi)).$$

Proof. The case when ϕ is an analytic diffeomorphism follows immediately from Proposition 2.3. This implies the case of open embedding. It is left to prove the case of linear projection $\phi : F^{n+m} \to F^n$. In this case the assertion follows from the fact that $\phi^*(\xi) = \xi \boxtimes 1_{F^m}$ where 1_{F^m} is the constant function 1 on F^m. □

This corollary enables to define the wave front set of any distribution on an F-analytic manifold, as a subset of the cotangent bundle. The precise definition follows.

Definition 2.5. *Let X be an F-analytic manifold and $\xi \in \mathcal{S}^*(X)$. We define the wave front set $WF(\xi)$ as the set of all $(x,\lambda) \in T^*X$ which lie in the wave front set of ξ in some local coordinates. In other words, $(x,\lambda) \in WF(\xi)$ if there exist open subsets $U \subset X$ and $V \subset F^n$, an analytic diffeomorphism $\phi : U \simeq V$ and $(y,\beta) \in T^*V$ such that $x \in U$, $\phi(x) = y, d_x\phi^*(\beta) = \lambda$, and $(y,\beta) \in WF((\phi^{-1})^*(\xi|_U))$.*

Theorem 2.6. *(Corollary from* [A13, Theorem 4.1.5]*) Let an F-analytic group H act on an F-analytic manifold Y and let χ be a character of H. Let $\xi \in \mathcal{S}^*(Y)^{(H,\chi)}$. Then*

$$WF(\xi) \subset \{(x,v) \in T^*Y | v(T_x(Hx)) = 0\}.$$

Theorem 2.7 ([A13, Theorem 4.1.2])**.** *Let $Y \subset X$ be F-analytic manifolds and let $y \in Y$. Let $\xi \in \mathcal{S}^*(X)$ and suppose that* $\mathrm{Supp}(\xi) \subset Y$. *Then $WF_y(\xi)$ is invariant with respect to shifts by the conormal space $CN^X_{Y,y}$.*

Corollary 2.8. *Let M be an F-analytic manifold and $N \subset M$ be a closed algebraic submanifold. Let ξ be a distribution on M supported in N. Suppose that for any $x \in N$, we have $CN^M_{N,x} \nsubseteq WF_x(\xi)$. Then $\xi = 0$.*

Proof. Suppose $\xi \neq 0$ and let $x \in \mathrm{Supp}(\xi)$. Then $(x,0) \in WF_x(\xi)$. But then from Theorem 2.7 we have $CN^M_{N,x} \subseteq WF_x(\xi)$ which contradicts our assumption on ξ. □

2.3. **Vanishing of equivariant distributions.** The following criterion for vanishing of equivariant distributions follows from [BZ76, §6] and [Ber83, §1.5].

Theorem 2.9 (Bernstein-Gelfand-Kazhdan-Zelevinsky)**.** *Let an algebraic group* **H** *act on an algebraic variety* **X***, both defined over F. Let χ be a character of H. Let $Z \subset X$ be a closed H-invariant subset. Suppose that for any $x \in Z$ we have*

$$\chi|_{H_x} \neq \Delta_H|_{H_x}\Delta^{-1}_{H_x},$$

where Δ_H and Δ_{H_x} denote the modular functions of the groups H and H_x. Then there are no non-zero (H,χ)-equivariant distributions on X supported in Z.

2.4. **Characters of unipotent groups.** The following lemma is standard.

Lemma 2.10. *Let* $\mathbf{V}$ *be a unipotent algebraic group defined over* F*, let* α *be a (locally constant, complex) character of* V *and* β *be a non-trivial character of* F*. Then there exists an algebraic group morphism* $\phi : \mathbf{V} \to \mathbb{G}_a$ *such that* $\alpha = \beta \circ \phi$*.*

For completeness we include a proof in Appendix A. In the case when $\mathbf{V}$ is a maximal unipotent subgroup of a reductive group and F is an arbitrary field (of an arbitrary characteristic) this lemma is [BH02, Theorem 4.1].

3. Proof of Theorem A

Lemma 3.1. *Let* $x \in G$*. Let* ξ *be a* (U,ψ)*-left equivariant and* (H,χ)*-right equivariant distribution on* G *such that* $WF(\xi) \subset G \times \mathcal{N}_{\mathfrak{g}^*}$*. Then* $WF_x(\xi) \subset CN^G_{BxH,x}$*.*

Proof. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus contained in B, and let $\mathfrak{h}, \mathfrak{u}$ be the Lie algebras of H, U respectively. Identify $T^*_x G$ with $\mathfrak{g}^*$ using the right multiplication by x^{-1}. We have $CN^G_{BxH,x} = (\mathfrak{t} + \mathfrak{u} + ad(x)\mathfrak{h})^\perp$. Since ξ is $\mathfrak{u}$-equivariant, by Theorem 2.6 we have $WF_x(\xi) \subset \mathfrak{u}^\perp$. Similarly, since ξ is $\mathfrak{h}$-equivariant on the right, we have $WF_x(\xi) \subset (ad(x)\mathfrak{h})^\perp$. By our assumption $WF_x(\xi) \subset \mathcal{N}_{\mathfrak{g}^*}$. Now, $\mathfrak{u}^\perp \cap \mathcal{N}_{\mathfrak{g}^*} = (\mathfrak{t} + \mathfrak{u})^\perp$ and thus

$$WF_x(\xi) \subset (ad(x)\mathfrak{h})^\perp \cap \mathfrak{u}^\perp \cap \mathcal{N}_{\mathfrak{g}^*} = (ad(x)\mathfrak{h})^\perp \cap (\mathfrak{u}+\mathfrak{t})^\perp = (\mathfrak{t}+\mathfrak{u}+ad(x)\mathfrak{h})^\perp = CN^G_{BxH,x}.$$

□

Now we would like to describe the structure of the varieties $\mathcal{O}_c$. For this we will use the following notation.

Notation 3.2. *For a* $B \times H$ *double coset* $\mathcal{O} = BxH \subset G$ *define*

$$\tilde{\mathcal{O}}_c = \bigcup_{\mathcal{O}' = ByH \subset (\mathbf{B}x\mathbf{H})(F)} \mathcal{O}'_c$$

Lemma 3.3. *For any double coset* $\mathcal{O} = BxH \subset G$ *there exists a closed algebraic subvariety* $\tilde{\mathbf{O}}_c \subset \mathbf{B}x\mathbf{H}$ *s.t.* $\tilde{\mathcal{O}}_c = \tilde{\mathbf{O}}_c(F)$*.*

Proof. Note that

$$\mathcal{O}_c = \left\{ x \in \mathcal{O} \,|\, \left.\psi^{x^{-1}}\right|_{H \cap U^{x^{-1}}} = \chi|_{H \cap U^{x^{-1}}} \right\}.$$

Let $\mathbf{H}_x := \mathbf{H} \cap \mathbf{U}^{x^{-1}}$. Since $\mathbf{U}$ is normal in $\mathbf{B}$, for any $y \in (\mathbf{B}x\mathbf{H})(F)$ we have $\mathbf{H}_x = \mathbf{H}_y$. Thus we will denote $\mathbf{H}_{\mathcal{O}} := \mathbf{H}_x$.

By Lemma 2.10 there exist an additive character β of F and algebraic group homomorphisms $\psi' : \mathbf{U} \to \mathbb{G}_a$, $\chi' : \mathbf{H}_{\mathcal{O}} \to \mathbb{G}_a$ such that $\psi = \beta \circ \psi'$ and $\chi|_{H_{\mathcal{O}}} = \beta \circ \chi'$. Let us show that

$$\tilde{\mathcal{O}}_c = \left\{ y \in (\mathbf{B}x\mathbf{H})(F) | \left.(\psi')^{y^{-1}}\right|_{\mathbf{H}_{\mathcal{O}}} = \chi' \right\}$$

Indeed, if $y \in \tilde{\mathcal{O}}_c$ then $\beta \circ \left.(\psi')^{y^{-1}}\right|_{H_{\mathcal{O}}} = \beta \circ \chi'$, hence $\beta \circ (\chi' - (\psi')^{y^{-1}}|_{\mathbf{H}_{\mathcal{O}}}) = 1$, thus $\chi' - (\psi')^{y^{-1}}|_{\mathbf{H}_{\mathcal{O}}}$ is bounded on $H_{\mathcal{O}}$, and thus $\chi' - (\psi')^{y^{-1}}|_{\mathbf{H}_{\mathcal{O}}}$ is trivial. We obtain

$$\tilde{\mathcal{O}}_c = \{ y \in (\mathbf{B}x\mathbf{H})(F) \,|\, \forall u \in H_{\mathcal{O}} \text{ we have } \psi'(yuy^{-1}) = \chi'(u) \},$$

which is clearly the set of F-points of a closed algebraic subvariety of $\mathbf{B}x\mathbf{H}$. □

Corollary 3.4.

(1) There exists a stratification of $\mathcal{O}_c$ into a union of smooth F-analytic locally closed submanifolds $\mathcal{O}_c^i$ s.t. $\bigcup_{i\leq i_0}\mathcal{O}_c^i$ is open in $\mathcal{O}_c$.

(2) Moreover, if $\mathcal{O}_c\neq\mathcal{O}$ then the dimensions of $\mathcal{O}_c^i$ are strictly smaller than the dimension of $\mathcal{O}_c$.

Proof of Theorem A. Suppose that there exists a non-zero right (U,ψ)-equivariant and left (H,χ)-equivariant distribution ξ supported on Z such that $WF(\xi)\subset G\times\mathcal{N}_{\mathfrak{g}^*}$.We decompose G into $B\times H$-double cosets and prove the required vanishing coset by coset. For a $B\times H$-double coset $\mathcal{O}\subset G$ define $\mathcal{O}_s:=\mathcal{O}\setminus\mathcal{O}_c$ and stratify $\mathcal{O}_c$, using Corollary 3.4, to a union of smooth locally closed F-analytic subvarieties $\mathcal{O}_c^i$. The collection

$$\{\mathcal{O}_c^i\,|\,\mathcal{O}\text{ is a }B\times H\text{-double coset}\}\cup\{\mathcal{O}_s\,|\,\mathcal{O}\text{ is a }B\times H\text{-double coset}\}$$

is a stratification of G. Order this collection to a sequence $\{S_i\}_{i=1}^N$ of smooth locally closed F-analytic submanifolds of G such that $U_k:=\bigcup_{i=1}^k S_i$ is open in G for any $1\leq k\leq N$. Let k be the maximal integer such that $\xi|_{U_{k-1}}=0$. Suppose $k\leq N$ and let $\eta:=\xi|_{U_k}$. Note that $\mathrm{Supp}(\eta)\subset S_k$. We will now show that $\eta=0$, which leads to a contradiction.

Case 1. $S_k=\mathcal{O}_s$ for some orbit $\mathcal{O}$. Then $\eta=0$ by Theorem 2.9 since η is $(U\times H,\psi\times\chi)$-equivariant.

Case 2. $S_k\subset\mathcal{O}=\mathcal{O}_c$ for some orbit $\mathcal{O}$. Then $S_k\subset G\setminus Z$ and $\eta=0$ by the conditions.

Case 3. $S_k\subset\mathcal{O}_c\subsetneq\mathcal{O}$ for some orbit $\mathcal{O}$. In this case, by Corollary 3.4, $\dim S_k<\dim\mathcal{O}$ and thus

$$CN^G_{S_k,x}\supsetneq CN^G_{\mathcal{O},x}.$$

By Lemma 3.1 we have, for any $x\in S_k$,

$$WF_x(\eta)\subset CN^G_{\mathcal{O},x}\text{ and thus }CN^G_{S_k,x}\not\subset WF_x(\eta).$$

By Corollary 2.8 this implies $\eta=0$.

□

4. Proof of Corollaries B and C

Let $\mathbf{U}'$ denote the derived group of $\mathbf{U}$.

Lemma 4.1. *Let $\overline{W}$ be the Weyl group of $\mathbf{G}$. Let $\overline{w}\in\overline{W}$ and let $w\in G$ be a representative of $\overline{w}$. Suppose that $w\mathbf{U}w^{-1}\cap\mathbf{U}\subset\mathbf{U}'$. Then $\overline{w}$ is the longest element in $\overline{W}$.*

Proof. Let $\mathfrak{u}$ be the Lie algebra of $\mathbf{U}$. On the level of Lie algebras the condition $wUw^{-1}\cap U\subset U'$ means that $(Ad(w)\mathfrak{u})\cap\mathfrak{u}\subset\mathfrak{u}'$. The algebra $\mathfrak{u}$ can be decomposed as

$$\mathfrak{u}=\bigoplus_{\alpha\in\Phi^+}\mathfrak{g}_\alpha.$$

It is easy to see that

$$(Ad(w)\mathfrak{u})\cap\mathfrak{u}=\sum_{\alpha\in\Phi^+,\overline{w}^{-1}(\alpha)\in\Phi^+}\mathfrak{g}_\alpha.$$

Let $\Delta \subset \Phi^+$ be the set of simple roots in Φ^+. Then from the condition of the lemma we obtain that $\overline{w}^{-1}(\Delta) \subset \Phi^-$, and as a consequence $\overline{w}^{-1}(\Phi^+) \subset \Phi^-$. Let $\overline{w}_0$ be the longest element in $\overline{W}$. Then $\overline{w}_0\overline{w}^{-1}(\Phi^+) \subset \Phi^+$. Since Φ^+ is a finite set and $\overline{w}_0\overline{w}^{-1}$ acts by an invertible linear transformation, we get $\overline{w}_0\overline{w}^{-1}(\Phi^+) = \Phi^+$. Since simple roots are the indecomposable ones, it follows that $\overline{w}_0\overline{w}^{-1}(\Delta) = \Delta$. This implies that $\overline{w}_0\overline{w}^{-1} = 1$ (see e.g. [Hum72, §10.3]), and thus $\overline{w}_0 = \overline{w}$. □

Corollary 4.2. *Let* $\mathbf{H}$ *be a reductive group. Assume* $\mathbf{G} = \mathbf{H} \times \mathbf{H}$ *and let* $\Delta\mathbf{H} \subset \mathbf{G}$ *be the diagonal copy of* $\mathbf{H}$. *Denote* $\mathbf{X} = \mathbf{G}/\Delta\mathbf{H}$ *and let* $x \in X$ *be such that* $\mathbf{U}_x \subset \mathbf{U}'$. *Then the orbit* $\mathbf{B}x$ *is open in* $\mathbf{X}$.

Proof. We can identify $\mathbf{X}$ with $\mathbf{H}$ using the projection on the first coordinate. We can assume that $\mathbf{B} = \mathbf{B_H} \times \mathbf{B_H}$ where $\mathbf{B_H}$ is a Borel subgroup of $\mathbf{H}$. Let $\overline{W}$ be the Weyl group of $\mathbf{H}$ and W be a set of its representatives. By the Bruhat decomposition,

$$\mathbf{H} = \bigsqcup_{w \in W} \mathbf{B_H} w \mathbf{B_H}$$

It is well-known that the only open $\mathbf{B_H} \times \mathbf{B_H}$ orbit in $\mathbf{H}$ is $\mathbf{B_H} w_0 \mathbf{B_H}$, where $w_0 \in W$ is the representative of the longest Weyl element. Let $w \in W$. Let $\mathbf{U_H}$ be the nilradical of $\mathbf{B_H}$. Then

$$\mathbf{U}_w = \{(u_1, u_2) | u_1 w u_2 = w, \;\; u_1, u_2 \in \mathbf{U_H}\},$$

and we see that for a pair $(u_1, u_2) \in \mathbf{U}_w$ we have $u_1 = w u_2 w^{-1} \in w\mathbf{U_H}w^{-1}$. Therefore,

$$\mathbf{U}_w \cong \mathbf{U_H} \cap w\mathbf{U_H}w^{-1}.$$

Let

$$R = \{x \in \mathbf{X} \, | \mathbf{U}_x \subset \mathbf{U}'\} = \{x \in \mathbf{H} | \mathbf{U_H} \cap x\mathbf{U_H}x^{-1} \subset \mathbf{U_H}' = [\mathbf{U_H}, \mathbf{U_H}]\},$$

and let $\mathbf{R}$ be the corresponding algebraic variety. Since $\mathbf{U}$ and $\mathbf{U}'$ are normal in $\mathbf{B}$, we obtain that $\mathbf{R}$ is $\mathbf{B}$-invariant. The corollary follows now from Lemma 4.1. □

Corollary 4.3. *Let* $\mathbf{H} \subset \mathbf{G}$ *be a subgroup of Galois type. Then for every non-open* B*-orbit* $\mathcal{O} \subset G/H$ *there exists* $y \in \mathcal{O}$ *such that* $\psi(U_y) \neq 1$.

Proof. Let $\mathcal{O} \subset G/H$ be a non-open B-orbit and $x \in \mathcal{O}$. Then the map $\mathbf{B} \to \mathbf{G}$ given by the action on x is not submersive and thus $\mathbf{B}x$ is not Zariski open in $\mathbf{G}/\mathbf{H}$. By Corollary 4.2 this implies $\mathbf{U}_x \not\subset \mathbf{U}'$. Thus, there exists a non-degenerate character φ of U such that $\varphi(U_x) \neq 1$. For a fixed $x \in \mathcal{O}$, the set of characters φ' of U such that $\varphi'(U_x) \neq 1$ is Zariski-open, thus dense in the l-space topology and thus intersects the B-orbit of ψ. Thus there exists $y \in Bx = \mathcal{O}$ such that $\psi(U_y) \neq 1$. □

Proof of Corollary B. By Theorem A it is enough to show that $S \subset Z$. Let $\mathcal{O} \subset S$ be a $B \times H$ double coset. Corollary 4.3 implies that there exists $x \in \mathcal{O}$ such that $\psi|_{U \cap H^x} \neq 1$. Since H^x is reductive and U is unipotent, we have $\chi^x|_{U \cap H^x} = 1$, and thus $\mathcal{O} \subset Z$. □

Proof of Corollary C. Define $\tilde{\mathbf{G}} = \mathbf{G} \times \mathbf{G}$, $\tilde{\mathbf{H}} = \Delta(\mathbf{G}) \subset \tilde{\mathbf{G}}$ and $\tilde{\mathbf{B}} = \mathbf{B} \times \mathbf{B}$. The non-degenerate characters ψ_1, ψ_2 define a non-degenerate character of the nilradical $\tilde{U}$ of $\tilde{B}$. Note that $\tilde{\mathbf{H}} \subset \tilde{\mathbf{G}}$ is a subgroup of Galois type and that $\tilde{G}/\tilde{H}$ is naturally isomorphic to G. Let η be the pull-back of ξ to $\tilde{G}$ under the projection $\tilde{G} \to \tilde{G}/\tilde{H} \cong G$. Then we have $\operatorname{Supp} \eta \subset \tilde{S}$, where $\tilde{S}$ is the union of all non-open $\tilde{B} \times \tilde{H}$-double cosets in $\tilde{G}$. Also, by Corollary 2.4 we have $WF(\eta) \subset \tilde{G} \times \mathcal{N}_{\tilde{\mathfrak{g}}^*}$. By Corollary B we obtain $\eta = 0$ and thus $\xi = 0$. $\square$

Remark 4.4. Corollary B can not be generalized literally to arbitrary symmetric pairs. The reason is that neither can Corollary 4.3. For example consider the pair $\mathbf{G} = \mathbf{GL}_{2n}, \mathbf{H} = \mathbf{GL_n} \times \mathbf{GL_n}$, where the involution is conjugation by the diagonal matrix with first n entries equal to 1 and others equal to -1. Let x be the coset of the permutation matrix given by the product of transpositions

$$\prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (2i+1, 2n-2i),$$

and let $\mathbf{B}$ consist of upper-triangular matrices. Then $\mathbf{U}_x \subset \mathbf{U}'$, while $\mathbf{B}x$ is of middle dimension in $\mathbf{G}/\mathbf{H}$. It can be shown that there exists a (U, ψ)-left equivariant, H-right invariant distribution ξ on G supported in BxH and satisfying $WF(\xi) \subset G \times \mathcal{N}_{\mathfrak{g}^*}$.

However, Corollary D(ii) might hold for any spherical subgroup $\mathbf{H}$. In fact, this is the case over the archimedean fields, see [AG, Corollary B].

Appendix A. Proof of Lemma 2.10

Lemma A.1. *Let $\mathfrak{v}$ be the (F-points of) the Lie algebra of $\mathbf{V}$. Then the exponential map* $\exp : \mathfrak{v} \to V$ *maps the commutant $[\mathfrak{v}, \mathfrak{v}]$ of $\mathfrak{v}$ onto the subgroup $[V, V]$ of V generated (set-theoretically) by all the commutators in V.*

Proof. Let $\mathfrak{v}_i$ be the sequence of subalgebras of $\mathfrak{v}$ defined by $\mathfrak{v}_0 := \mathfrak{v}$, $\mathfrak{v}_{i+1} := [\mathfrak{v}, \mathfrak{v}_i]$. The Baker-Campbell-Hausdorff formula implies that for any $X \in \mathfrak{v}$ and $Y \in \mathfrak{v}_i$ there exist $A, B \in \mathfrak{v}_{i+2}$ and $C \in \mathfrak{v}_{i+1}$ such that

$$\log(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + A, \tag{1}$$

$$\log(e^X e^Y e^{-X} e^{-Y}) = [X, Y] + B \tag{2}$$

$$e^{X+Y} = e^C e^X e^Y. \tag{3}$$

By (1,2) we have $[V, V] \subset \exp([\mathfrak{v}, \mathfrak{v}])$. To prove the opposite inclusion we prove by descending induction on i that $\exp(\mathfrak{v}_i) \subset [V, V]$ for any $i > 0$. Since $\exp(\mathfrak{v}_i)$ is a group, it is enough to show that for any $X \in \mathfrak{v}$ and $Y \in \mathfrak{v}_{i-1}$ we have $\exp([X, Y]) \in [V, V]$. Let B be as in (2), and C be as in (3) applied to $\log(e^X e^Y e^{-X} e^{-Y})$ and $-B$. Then $B, C \in \mathfrak{v}_{i+1}$, the induction hypothesis implies that $e^B, e^C \in [V, V]$ and thus

$$\exp([X, Y]) = \exp(\log(e^X e^Y e^{-X} e^{-Y}) - B) = e^C (e^X e^Y e^{-X} e^{-Y}) e^{-B} \in [V, V].$$

$\square$

Corollary A.2. *Let $\mathbf{V}/[\mathbf{V}, \mathbf{V}]$ denote the abelization of $\mathbf{V}$. Then the natural map $V/[V, V] \to (\mathbf{V}/[\mathbf{V}, \mathbf{V}])(F)$ is an isomorphism.*

Proof. Let $\underline{\mathfrak{v}}$ be $\mathfrak{v}$ considered as an algebraic variety. By (1,2), the quotient $\mathbf{V}/\exp([\underline{\mathfrak{v}}, \underline{\mathfrak{v}}])$ is an abelian group. Hence $[\mathbf{V}, \mathbf{V}] \subset \exp([\underline{\mathfrak{v}}, \underline{\mathfrak{v}}])$. Thus, by Lemma A.1 we have $[V, V] \subset [\mathbf{V}, \mathbf{V}](F) \subset \exp([\mathfrak{v}, \mathfrak{v}]) = [V, V]$. Therefore $[V, V] = [\mathbf{V}, \mathbf{V}](F)$. Since unipotent groups have trivial Galois cohomologies (see [Ser97, §III.2.1, Proposition 6]), $\mathbf{V}(F)/[\mathbf{V}, \mathbf{V}](F) = (\mathbf{V}/[\mathbf{V}, \mathbf{V}])(F)$ and the statement follows. $\square$

By this corollary Lemma 2.10 reduces to the case when $\mathbf{V}$ is commutative. Since any commutative unipotent group over F is a power of $\mathbb{G}_a$, this case follows from the isomorphism of F to its Pontryagin dual. $\square$

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