Course on p-adic Analysis.
Problem assignment 1.

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Definition. Let Z be a topological space and F a sheaf of abelian groups on Z. We say that F is **Γ-acyclic** if $H^i(X, F) = 0$ for all $i > 0$.

Example. Let X be an affine algebraic variety over some field K and F a quasi-coherent sheaf of O_X-modules on X. Then, according to Serre's theorem, the sheaf F is Γ-acyclic as a sheaf of abelian groups.

1. Let Z be a topological space, $F \in Sh(Z)$ a sheaf of abelian groups on Z.

Show that one can compute the cohomology $H^*(Z, F)$ using Γ-acyclic resolutions instead of injective resolutions.

Show the same for any bounded bellow complex $\mathcal{N} \in Com^+(Sh(Z))$.

2. Let X be a separated algebraic variety over a field K (you can assume X to be quasi-projective). Consider a finite covering $\mathcal{U} = (U_i)$ of X by affine open subsets.

(i) Let F be a quasi-coherent sheaf of O_X-modules on X. Show how to explicitly compute the cohomology $H^*(X, F)$ using the $\hat{\text{C}}$ech complex $H_{\mathcal{C}}(X, F)$.

(ii) Do the same for the hyper-cohomology of a bounded bellow complex $\mathcal{N}$ of quasi-coherent O_X-modules.

3. Let X be a smooth variety over a field K.

(i) Using the $\hat{\text{C}}$ech method show how explicitly compute the DeRham cohomology $H_{DR}(X)$.

(ii) Let L/K be a field extension. Denote by X_L the algebraic variety over L obtained from X by extension of scalars.

Show that DeRham cohomology $H^*_{DR}(X_L)$ is canonically isomorphic to $L \otimes_K H^*_{DR}(X)$.

4. Show in detail how GAGA theorem by Serre implies that for a smooth projective variety X over $\mathbb{C}$ the DeRham cohomology are canonically isomorphic to $H^*(X_{an}, \mathbb{C})$.

5. Let Y be the projective line defined over the field $\mathbb{Q}$, $X = Y_{\mathbb{C}}$ be its extension of scalars to $\mathbb{C}$.

We have seen that the one dimensional spaces $H^2(X_{an}, \mathbb{C})$ and $H^2_{DR}(X)$ are canonically isomorphic. Both of these spaces have natural $\mathbb{Q}$ structures. Show that these structures differ by multiplication by the number $2\pi i$. In particular, there is no natural identification of $\mathbb{Q}$-vector spaces $H^2(Y, \mathbb{Q})$ and $H^2_{DR}(Y)$.

(**Hint.** Show that the space $H^2(X)$ is canonically identified with the space $H^1(G_m)$).

6. Show that there exists a subfield $L \subset \mathbb{C}$ of countable transcendence degree over $\mathbb{Q}$ such that for all algebraic varieties defined over $\mathbb{Q}$ the isomorphism of Betti and DeRham cohomologies preserves L-structures.

7. Show that the Hodge Theorem is equivalent to the following

Statement. For every integer n we have
$\dim H^n_{DR}(X) = \sum_{p+q=n} \dim H^p(X, \Omega^q)$.

8. Show explicitly that Hodge Theorem over $\mathbb{C}$ implies Hodge theorem over any field K of characteristic 0.