HD expanders

15/10/2013

The story starts w a thm proved by two high-school students

Theorem (Boros-Fured, 80's) Given n pts $x_1, \dots x_n \in \mathbb{R}^2$

There exists a pt $z \in \mathbb{R}^2$ which is covered by at least $\left(\frac{2}{9} - \bar{o}(1)\right)\binom{n}{3}$ of the $\binom{n}{3}$ triangles determined by $\{x_1, \dots x_n\}$.

Theorem (Barany 80) Fix $d \geq 2$, $\exists\, c_d > 0$ s.t. for every $P = \{x_1, \dots x_n\} \in \mathbb{R}^d$ $\exists\, z \in \mathbb{R}^d$ which is contained at at least $c_d \binom{n}{d+1}$ of the simplicies determined by P.

For $d=2$, $c_d = \frac{2}{9}$ is optimal. (result of someone...)

$\exists$ some estimates on c_d are known, but not optimal.

[Gromov (GAFA, 2010)

Def A finite simp. complex X is a ~~subset~~ collection of subsets of a set $V = X(0)$ (= vertices of X), s.t. if $F \in X$ and $G \subset F$ then $G \in X$.

If $F \in X$ and $|F| = i+1$, say $\dim F = i$

Note $\emptyset \in X$, $\dim \emptyset = -1$.

F is a face of dim i, "i-face"

$$X(i) = \text{the set of } i\text{-faces}$$

X is of dim d, if $X(d) \neq \emptyset$, but $X(n) = \emptyset$, $\forall n > d$

Assume $\dim X = d$ & $X(0) \to \mathbb{R}^d$ we can extend f linearly (= affinely) $\tilde{f} : X \to \mathbb{R}^d$

Def X - s.c. of dim d, and $\varepsilon > 0$ we say that X has the ε-geometric overlapping property, if $\forall f : X(0) \to \mathbb{R}^d$, there exists a pt $z \in \mathbb{R}^d$ s.t.

$\tilde{f}^{-1}(z)$ intersects at least ε-fraction of the d-faces of X.

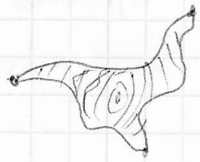
$\binom{n}{3} = \frac{n!}{(n-3)!\,3!} = \frac{n(n-1)(n-2)}{6}$

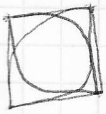
Let $\Delta_n^{(d)}$ = the complete d-dim complex on n vertices = all the subsets of size $\leq d+1$ of $X(0) = \{1, \dots n\}$

Barany thm says that $\exists$ $c_d > 0$ s.t. every $\Delta_n^{(d)}$ (for all n) has the c_d-geom over. property.

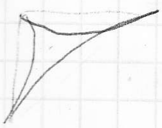
~~Theorem~~

Def X is a s.c. of dim d, $\varepsilon > 0$. We say that X has the ε-topological overlapping property if $\forall$ $f : X(0) \to \mathbb{R}^d$ and $\forall$ $\bar{f} : X \to \mathbb{R}^d$ extending f continously

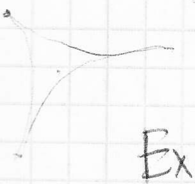




$\exists$ $z \in \mathbb{R}^d$ s.t. $\bar{f}^{-1}(z)$ intersects at least ε-fraction of the d-faces of X.



Theorem (Gromov) $\exists$ $c_d' > 0$, $\Delta_n^{(d)}$ has the c_d'-topological overlapping property.



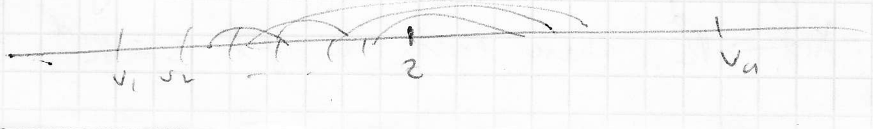
Expander graphs

X-graph, i.e. a s.c. of dim $= 1$, $\varepsilon > 0$.

X is called an ε-expander, if $\forall$ $Y \subseteq X(0)$

$$|E(Y, \bar{Y})| \geq \varepsilon \cdot \min(|Y|, |\bar{Y}|)$$

If X is an ε-expander, $f : X(0) \to \mathbb{R}^1$ there is a pt $z \in \mathbb{R}$ s.t. half of $f(X(0))$ are "below" z and "half are above"



Linial - Meshulam model of random complex.

↳ Erdős-Rényi model of random graph!

Take n points, $\forall$ 2 vertices i, j:

$$P(\,(i,j) \text{ is an edge}\,) = p.$$

$\frac{2}{9}\binom{n}{3} < 25^6$

$\frac{n(n-1)(n-3)}{6} < 9 \cdot 128$

$n(n-1)(n-3) < \frac{27 \cdot 25^6}{3^?\cdot 3^?}$

$n \approx 24$

$X(n,p)$ - "the" random graph on n vertices with every edge appearing w/ prob. p.

<u>Thm</u> If $p < \frac{\log n}{n} - \varepsilon$ then $X(n,p)$ is a.s. not connected

If $p > \frac{\log n}{n} + \varepsilon$, then a.s. connected.

L-M

↳ Take n pts, take <u>all</u> pairs, i.e. all edges.

$X_2(n,p)$ = each tr. is taken ind. w/ prob. p.

When $H_1(X_2(n,p), \mathbb{F}_2) = 0$?

<u>Thm</u> If. $p < \frac{2\log n}{n} - \varepsilon \Rightarrow$ a.s. $\neq 0$

$p > \frac{2\log n}{n} + \varepsilon \Rightarrow$ a.s. $= 0$

Meshulam - Wallach

d-dim case.

— " $< d \cdot \ldots \Rightarrow H_{d-1}(\ldots) \neq 0$

$> d \cdot \ldots \Rightarrow \qquad = 0$

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One-dimensional expanders (expander graphs)

<u>Def</u> Graph = 1-dim complex, $X = G = (V, E)$

$X(0) = V$, $X(1) = E$

X - graph, the Cheeger constant:

$$h(X) = \min_{\substack{A \sqcup B = \\ = V}} \frac{|E(A,B)|}{\min\{|A|, |B|\}}$$

X - is an ε-expander $\Longleftrightarrow h(X) \geq \varepsilon$