Linial - Meshulam model of random complex.

↳ Erdős-Renyi model of random graph!

Take n points, $\forall$ 2 vertices i,j:

$P((i,j)$ is an edge$) = p$.

$\frac{2}{9}\binom{n}{3} < 2^{56}$

$X(n,p)$ - "the" random graph on n vertices with every edge appearing w/ prob. p.

$\frac{n(n-1)(n-3)}{6} < 9 \cdot 12^3$

$\frac{n(n-1)(n-3)}{3 \cdot 3} < 27 \cdot 2^{56}$

$n = 24$

<u>Thm</u> If $p < \frac{\log n}{n} - \varepsilon$ then $X(n,p)$ is a.s. not connected

If $p > \frac{\log n}{n} + \varepsilon$, then a.s. connected.

L-M

↳ Take n pts, take <u>all</u> pairs, i.e. all edges.

$X_2(n,p)$ = each tr. is taken ind. w/ prob. p.

When $H_1(X_2(n,p), \mathbb{F}_2) = 0$?

<u>Thm</u> If $p < \frac{2\log n}{n} - \varepsilon \Rightarrow$ a.s. $\neq 0$

$p > \frac{2\log n}{n} + \varepsilon \Rightarrow$ a.s. $= 0$

Meshulam - Wallach

d-dim case

$-\text{''} < d \cdot \ldots \Rightarrow H_{d-1}(\ldots) \neq 0$

$> d \cdot \ldots \Rightarrow = 0$

Uriya First. 22/10

One-dimensional expanders (expander graphs)

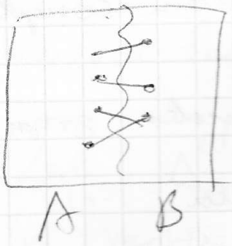
<u>Def</u> Graph = 1-dim complex, $X = G = (V, E)$

$X(0) = V$, $X(1) = E$

X - graph, the Cheeger constant:

$$h(X) = \min_{\substack{A \sqcup B = \\ = V}} \frac{|E(A,B)|}{\min\{|A|, |B|\}}$$

X - is an ε-expander $\Longleftrightarrow h(X) \geq \varepsilon$



prc.

Dfn the adj matrix of X:

$$A = (a_{ij})_{i,j\in V} \qquad a_{ij} = \#\{\text{edges from } i \text{ to } j\}$$

A is symmetric $\Rightarrow$ A is diagonalizable with real spectrum

$$-k \le \lambda_0 \le \lambda_1 \le \dots \le \lambda_{n-1} = k$$

if X is k-regular   if X is k-regular

Remark when X is k-regular λ_0, λ_{n-2} control how fast a random walk on X converges to the uniform distribution on V

P - prob on the vertices of X

$$p \xrightarrow{\text{one step}} \frac{1}{k} A p$$

$$p = \sum \alpha_i v_i, \text{ where } A v_i = \lambda_i v_i$$

$$\left(\frac{1}{k} A\right)^r p = \frac{1}{k^r} \sum \alpha_i \lambda_i^r v_i \sim k^{-r} \lambda_{n-1}^r v_{n-1}$$

$$(|\lambda_{n-2}|, |\lambda_0| < \lambda_{n-1})$$

since $\left(\frac{\lambda_i}{k}\right)^r \le 1$

this is a constant vector

Def The Laplacian

Let $C_i(X) = L_2(X(i))$ $i = 0, 1$

Hilbert space

Fix an order $v_1 \le v_2 \le \dots \le v_n$ on V

Define $\delta_0 : C_0(X) \to C_1(X)$

$$(\delta \varphi)(e) = \varphi(e^+) - \varphi(e^-), \text{ where } e^- \le e^+$$

We also have a dual map

$$C_1(X) \xrightarrow{\partial = \delta^*} C_0(X)$$

$$(\delta^* \varphi)(v) = \sum_{\substack{e \\ v=e^+}} f(e) - \sum_{\substack{e \\ v=e^-}} f(e)$$

The Laplacian of X is

$\Delta = \Delta_0^+ = \delta^* \delta : C_0(X) \to C_0(X)$

<u>Claim</u>: Δ is independent of the ordering on V

<u>Proof</u> Let us compute:

$$(\Delta\varphi)_v = \delta^*\delta(\varphi)(v) = \sum_{\substack{e \\ v=e^+}} (\delta\varphi)e - \sum_{\substack{e \\ v=e^-}} (\delta\varphi)e =$$

$$= \sum_{\substack{e \\ v=e^+}} (\underbrace{\varphi(e^+)}_{\varphi(v)} - \varphi(e^-)) - \sum_{\substack{e \\ v=e^-}} (\varphi(e^+) - \underbrace{\varphi(e^-)}_{\varphi(v)}) =$$

$$= \sum_{\substack{e \\ v\in e}} \varphi(v) - \varphi(v \setminus \{e\})$$ ← clearly indep. of the ordering on V.

$\boxed{}$

$$(\Delta\varphi)v = \sum_{\substack{u \in V \\ u \sim v}} (\varphi(v) - \varphi(u)) = \deg(v) \cdot \varphi(v) - (A\varphi)(v)$$

Since $\Delta = \delta^*\delta$, Δ is symm. and positive definite
$\Rightarrow$ diagonalizable w/ non-negative real spectrum

$$0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}$$

<u>Dfn</u> $\lambda(X) = \lambda_1(\Delta)$ is called the spectral gap of X.

<u>Note</u> when X is k-regular, $\lambda(X) = \lambda_{n-1}(A) - \lambda_{n-2}(A)$

<u>Thm</u> $\frac{h(X)^2}{2m} \leq \lambda(X) \leq 2h(X)$, $m = \max_v \deg v$

<u>Informally</u>: being a good expander $\sim$ having large spectral gap.

Lemma $L_2^0(V) = \{\varphi \in L_2(V) : \sum \varphi(v) = 0\}$

Then $\lambda(X) = \inf_{0 \neq \varphi \in L_2^0(V)} \frac{\|\delta \varphi\|_2^2}{\|\varphi\|_2^2}$

Proof w.l.o.g. assume $\|\varphi\|_2 = 1$

$\|\delta\varphi\|^2 = \langle \delta\varphi, \delta\varphi \rangle =$ (inner prod of $L_2(V)$) $=$

$= \langle \varphi, \delta^* \delta\varphi \rangle = \langle \varphi, \Delta\varphi \rangle$

The inf of $\langle \varphi, \Delta\varphi \rangle$ is the lowest e.v. of Δ on $L_2^0(V)$ which is $\lambda_1(\Delta) = \lambda(X)$ $\square$

Explanation of (*) : Δ is self-adjoint, diagonalizable

$\Rightarrow$ $L_2^0(V)$ has orthonormal basis $\varphi_1, \dots \varphi_{n-1}$

with $\Delta\varphi_i = \lambda_i \varphi_i$

write $\varphi = \sum_{i=1}^{n-1} \alpha_i \varphi_i$

$\langle \varphi, \Delta\varphi \rangle = \langle \sum \alpha_i \varphi_i, \sum \alpha_i \lambda_i \varphi_i \rangle =$

$= \sum \alpha_i^2 \lambda_i \langle \varphi_i, \varphi_i \rangle = \sum_{i=1}^{n-1} \alpha_i^2 \lambda_i = \star$

(keep in mind: $\sum \alpha_i^2 = 1$)

$\inf \star = \lambda_1.$

$\square$

Proof of $(\lambda(X) \leq 2h(X))$:

Recall $h(X) = \min_{A \sqcup B = V} \frac{|E(A,B)|}{\min\{|A|, |B|\}}$,

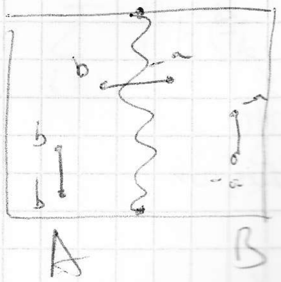
let $V = A \cup B$, $a = |A|$, $b = |B|$, w.l.o.g $a \leq b$

Define $\varphi \in L_2^0(V)$ by

$\varphi(v) = \begin{cases} b & v \in A \\ -a & v \in B \end{cases}$

Compute $\frac{\|\delta\varphi\|^2}{\|\varphi\|^2} = \frac{\sum_e |\delta\varphi(e)|^2}{\sum_v |\varphi(v)|^2} = \frac{\sum_e |\varphi(e^+) - \varphi(e^-)|^2}{a \cdot b^2 + b a^2} =$

$= \frac{n^2 \cdot \#\{\text{edges from } A \text{ to } B\}}{nab} = \frac{n|E(A,B)|}{a \cdot b} = \frac{n E(A,B)}{a \cdot b} \leq$



$$\leq \frac{n\,|E(A,B)|}{a\cdot\binom{n}{2}} \leq 2\cdot\frac{|E(A,B)|}{|A|} =$$

since $\frac{n}{b} \leq \frac{1}{2}$

$$= 2\cdot\frac{|E(A,B)|}{\min\{|A|,|B|\}}$$

This is true for all partitions $V = A \sqcup B$, so

$$\lambda(X) = \inf \frac{\|\delta\varphi\|}{\|\varphi\|^2} \leq 2h(X)$$

<u>Proof of (2):</u>

Let $g \in L_2^0(V)$ be the e.f. of Δ with e.v $\lambda_1(\Delta)$ ~~and~~ with $\|g\| = 1$.

Define

$$f = \max\{g, 0\},\quad f(v) = \begin{cases} g(v) & , g(v) \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$V^+ = \{v \in V \mid g(v) > 0\}$. w.l.o.g. $|V^+| \leq \frac{1}{2}|V|$, otherwise replace g with $-g$.

$$\|\delta f\|^2 = \langle \delta f, \delta f\rangle = \langle f, \Delta f\rangle = \sum_{v\in V} f(v)\,\Delta f(v) =$$

$$= \sum_{v\in V^+} g(v)\,\Delta f(v) = \sum_{v \in V^+} g(v) \sum_{v\sim u\in V} (\underbrace{f(v)}_{=g(v)} - \underbrace{f(u)}_{\geq g(u)}) \leq$$

$$\leq \sum_{v\in V^+} g(v) \sum_{u \in V} (g(v) - g(u)) = \sum_{v\in V^+} g(v)\,\Delta g(v)$$

$$= \sum_{v\in V^+} g(v)^2 \lambda_1 = \lambda_1 \|f\|^2$$

Conclusion I: $\|\delta f\|^2 \leq \lambda_1 \|f\|^2$

$$(A) := \sum_{e\in E} |f(e^+)^2 - f(e^-)^2| =$$

$$= \sum_e |f(e^+) + f(e^-)|\cdot|f(e^+) - f(e^-)| \leq$$

$$\leq \left(\sum_e |f(e^+) + f(e^-)|^2\right)^{\frac{1}{2}} \left(\sum_e |f(e^+) - f(e^-)|^2\right)^{\frac{1}{2}} =$$

$$\leq \Big(\sum_e |2 f(e^+)^2 + 2 f(e^-)^2| \Big)^{\frac{1}{2}} \cdot \| \delta f \| \leq$$

since $(a+b)^2 \leq 2a^2 + 2b^2$

by the conclusion

$$\leq \sqrt{2} \Big(\sum f(e^+)^2 + f(e^-)^2 \Big)^{1/2} \cdot \| \delta f \| \leq$$

$$\leq \sqrt{2} \Big(m \cdot \sum_{v \in V} f(v)^2 \Big)^{1/2} \cdot \| \delta f \|^2 = \sqrt{2m}\, \|f\| \cdot \|\delta f\| \leq$$

$$\leq \sqrt{2m} \cdot \sqrt{\lambda_1} \cdot \|f\|^2$$

<u>Conclusion II</u>: $(A) \leq \sqrt{2m\lambda_1} \cdot \|f\|^2$

Let $0 = \beta_0 < \beta_1 < \dots < \beta_r$ be the values of f

Let $L_i = \{ v \in V \mid f(v) \geq \beta_i \}$

$$V = L_0 \supsetneq L_1 \supset L_2 \supset \dots \supset L_r$$

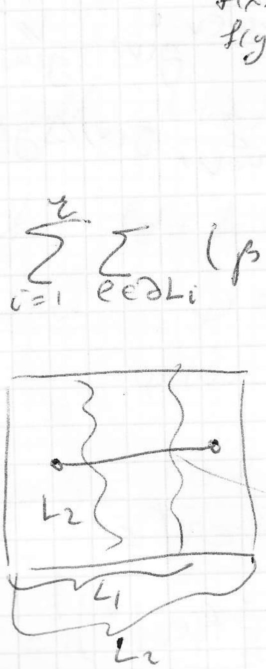
$|L_1| \leq \frac{1}{2}|V|$, since $L_1 = V^+$

$$(A) = \sum_{e \in E} | f(e^+)^2 - f(e^-)^2 | =$$

$$= \sum_{i=1}^{r} \sum_{\substack{x \sim y \\ f(x) = \beta_i \\ f(y) < \beta_i}} \big(f(x)^2 - f(y)^2 \big) =$$

$$\beta_i^2 - \beta_j^2 = (\beta_i^2 - \beta_{i-1}^2) + (\beta_{i-1}^2 - \beta_{i-2}^2) + \dots + (\beta_{j+1}^2 - \beta_j^2)$$

$$= \sum_{i=1}^{r} \sum_{e \in \partial L_i} (\beta_i^2 - \beta_{i-1}^2)$$



contributes $\beta_1^2 - \beta_0^2$ to $\sum_{e \in \partial L_1}$

$\beta_2^2 - \beta_1^2$ to $\sum_{e \in \partial L_2}$

$$= \sum_{i=1}^{r} |E(L_i, V \setminus L_i)| \cdot (\beta_i^2 - \beta_{i-1}^2) \geq$$

$$\geq \sum_{i=1}^{r} h(X) \cdot \underset{\substack{| \\ \text{since} \\ |L_i| \leq |L_1| \leq \frac{1}{2}|V|}}{|L_i|} \cdot |\beta_i^2 - \beta_{i+1}^2| =$$

$$= h(X) \Big| |L_r| \cdot \beta_r^2 + \sum_{i=1} \beta_i^2 \cdot \big(\underset{\substack{\| \\ \#\{v \in V / f(v) = \beta_i\}}}{|L_i| - |L_{i+1}|}\big) =$$

$$= h(X) \sum_{v \in V} f^2(v) = h(X) \cdot \|f\|^2 \leftarrow \text{conclusion III}$$

$$\Rightarrow \big(\underset{\text{conclusion}}{\mathrm{II} + \mathrm{III}}\big) \Rightarrow \quad \|f\|^2 h(X) \leq \sqrt{2m\lambda_1} \cdot \|f\|^2$$

$$\Downarrow$$

$$\lambda_1(X) \geq \frac{h(X)^2}{2m}. \quad \boxed{}$$

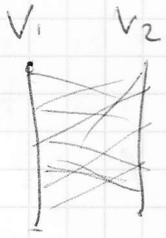
29/10/13

Lubotzky

X - a finite graph $A = A_X =$ adj matrix of A

If X is k-regular then k is the largest e.v. with mult 1 iff X is connected and $-k$ is an e.v. $\Leftrightarrow$ X is bipartite.

$\Leftarrow$



$$|E| = |V_1| \cdot k = |V_2| k \Rightarrow |V_1| = |V_2|$$

$$f(x) = \begin{cases} 1 & x \in V_1 \\ -1 & x \in V_2 \end{cases} \Rightarrow Af = (-k) f$$

Moreover, in this case the spec(A) is sym, i.e. if $\lambda \in$ Spec then $-\lambda \in$ Spec, because if $f \in L^2(V)$ with $Af = \lambda f$, then

$$\bar{f}(x) = \begin{cases} f(x) & x \in V_1 \\ -f(x), & x \in V_2 \end{cases}$$

$$A\bar{f} = (-\lambda)\bar{f}$$