$$= \sum_{i=1}^{r} |E(L_i, V \setminus L_i)| \cdot (\beta_i^2 - \beta_{i-1}^2)^2 \geq$$

$$\geq \sum_{i=1}^{r} h(x) \cdot \underset{\substack{|\\ \text{since} \\ |L_i| \leq |L_1| \leq \frac{1}{2}|V|}}{|L_i|} \cdot |\beta_i^2 - \beta_{i+1}^2| =$$

$$= h(x)\left[|L_r| \cdot \beta_r^2 + \sum_{i=1}^{r} \beta_i^2 \cdot \underbrace{(|L_i| - |L_{i+1}|)}_{\#\{v \in V / f(v) = \beta_i\}}\right] =$$

$$= h(X) \sum_{v \in V} f^2(v) = h(X) \cdot \|f\|^2 \Leftarrow \text{conclusion III}$$

$$\Rightarrow \left(\text{II} + \underset{\text{conclusion}}{\text{III}}\right) \Rightarrow \quad \|f\|^2 h(X) \leq \sqrt{2m\lambda_1} \cdot \|f\|^2$$

$$\Downarrow$$

$$\lambda_1(X) \geq \frac{h(X)^2}{2m}. \quad \boxtimes$$

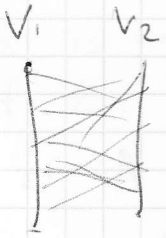
29/10/13

Lubotzky

X - a finite graph $A = A_X =$ adj matrix of A

If X is k-regular then k is the largest e.v. with mult 1 iff X is connected and $-k$ is an e.v. $\Leftrightarrow$ X is bipartite.

$\Leftarrow$



$$|E| = |V_1| \cdot k = |V_2| k \Rightarrow |V_1| = |V_2|$$

$$f(x) = \begin{cases} 1 & x \in V_1 \\ -1 & x \in V_2 \end{cases} \Rightarrow Af = (-k) f$$

Moreover, in this case the spec(A) is sym, i.e if $\lambda \in$ Spec then $-\lambda \in$ Spec, because if $f \in L^2(V)$ with $Af = \lambda f$, then

$$\bar{f}(x) = \begin{cases} f(x) & x \in V_1 \\ -f(x), & x \in V_2 \end{cases}$$

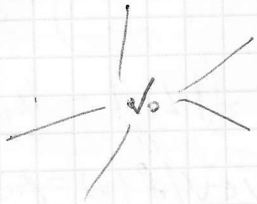
$$A\bar{f} = (-\lambda)\bar{f}$$

$\Rightarrow$ **Assume** $-k$ is an e.v. w/ f as e.f.

Define : ~~$V_1 = \{v \in V \mid f(v) > 0\}$~~

v_0 is a vertex with $|f(v_0)|$ maximal

w.l.o.g. we can assu $f(v_0) > 0$



$$(Af)(v_0) = \sum_{y \sim v_0} f(y) = -kf(v_0)$$

$$|Af(v_0)| \leq k \cdot f(v_0) \Rightarrow$$

$$\Rightarrow |f(y)| = |f(v_0)| \; \forall \; y \sim v_0$$

$$\Downarrow$$

$$f(y) = -f(v_0)$$

continue like that and get that

$$V = \underset{\overset{\in}{v_0}}{V_1} \sqcup V_2 : \; ~~\forall v \in V~~ \; f|_{V_1} \equiv f(v_0)$$

$$f|_{V_2} \equiv -f(v_0)$$

and there are no edges between V_1 and V_2.

Assume now X is a (k_1, k_2) bi-regular bi-partite graph : $V = V_1 \sqcup V_2$, $\deg(V_1) = k_1$

$\deg(V_2) = k_2$

$$k_1 \cdot |V_1| = k_2 \cdot |V_2|$$

$$A = \mathrm{Adj}(X)$$

Claim 1) The largest e.v. of A is $\sqrt{k_1 k_2}$

2) $\mathrm{Spec}(A)$ is sym (λ e.v. $\Leftrightarrow$ $-\lambda$ e.v

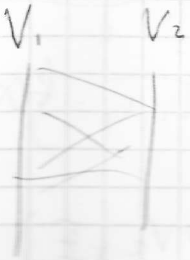
3) The e.f. of $\sqrt{k_1 k_2}$ is $f(x) = \begin{cases} \frac{1}{\sqrt{k_2}} & x \in V_1 \\ \frac{1}{\sqrt{k_1}} & x \in V_2 \end{cases}$

Remark X - any graph , $A = A_X$ - adj.

$$(A_X^\ell)_{ij} = \#\{\text{paths from } i \text{ to } j \text{ in } A \text{ of length } \ell\}$$

Proof: exercise

Back to the claim



$$A = \left(\begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array}\right), B = |V_1| \times |V_2|$$

$$A^2 = \begin{pmatrix} BB^t & 0 \\ 0 & B^tB \end{pmatrix}$$

$$\mathrm{diag}(A^2) = (\underbrace{k_1, \dots k_1}_{|V_1|}, \underbrace{k_2, \dots k_2}_{|V_2|})$$

Lemma BB^t and B^tB have the same e.v. up to mult. of 0's

Pf: if α e-vector: $(B^tB)\alpha = \lambda\alpha \Rightarrow$

$\Rightarrow (BB^t)(B\alpha) = \lambda(B\alpha) \Rightarrow$

$\Rightarrow$ $B\alpha$ is e.v. of BB^t w/ the same e.v. (if $B\alpha \neq 0$)

But if $B\alpha = 0 \Rightarrow \lambda = 0$. ☒

A^2 is ~~the~~ adj matrix + diag. of ~~a~~ graph w/ ~~tho~~ two connected components.

k_1k_2 is an e.v. of A^2 with $f = 1$ with mult 2:

$$f_1 = \begin{cases} 1 & \text{on } V_1 \\ 0 & \text{on } V_2 \end{cases}$$

$$f_2 = \begin{cases} 0 & \text{on } V_1 \\ 1 & \text{on } V_2 \end{cases}$$

$A^2 \cdot f_i = k_1k_2 f_i$

Therefore the largest e.v. of A is ~~at most~~ $\sqrt{k_1k_2}$

Pf (2) if f is an e.f. of A with e.v λ:

$$\Rightarrow \bar{f}(x) = \begin{cases} f(x) & x \in V_1 \\ -f(x) & x \in V_2 \end{cases} \Rightarrow A\bar{f} = -\lambda\bar{f}$$ ☒

↳

3) $\underset{x \in V_1}{(Af)(x)} = k_1 \cdot \frac{1}{\sqrt{k_1}} \overset{?}{=} \underbrace{\sqrt{k_1 k_2} \cdot \frac{1}{\sqrt{k_2}}}_{\sqrt{k_1k_2} \cdot f(x)} = \sqrt{k_1}$

✓

$\square$

<u>Mixing lemma</u> $X = (V_1 \cup V_2, E)$ bi-partite.

bi-reg (k_1, k_2), connected, Denote

$$\lambda = \max \left\{ \lambda \;\middle|\; \begin{array}{l} \lambda \text{ e.v of Adj} \\ \lambda \neq \sqrt{k_1k_2} \end{array} \right\}$$

If $A \subseteq V_1$, $B \subseteq V_2$

$$\left| |E(A,B)| - \sqrt{k_1k_2} \cdot \frac{|A| \cdot |B|}{\sqrt{|V_1||V_2|}} \right| \leq \lambda \sqrt{|A| \cdot |B|}$$

i.e. note $k_1 \cdot |V_1| = k_2 |V_2| =: |E| \Rightarrow$

$$\Rightarrow \sqrt{\frac{k_1 k_2}{|V_1||V_2|}} = \sqrt{\frac{|E|^2}{|V_1|^2 |V_2|^2}} = \frac{|E|}{|V_1| \cdot |V_2|}$$

$$\left| \frac{|E(A,B)|}{|E|} - \frac{|A| \cdot |B|}{|V_1| \cdot |V_2|} \right| \leq \lambda \sqrt{\frac{|A|}{|E|}} \sqrt{\frac{|B|}{|E|}} =$$

number of expected edges: $= \frac{\lambda}{\sqrt{k_1k_2}} \sqrt{\frac{|A|}{|V_1|} \frac{|B|}{|V_2|}}$

$$|E| \frac{|A||B|}{|V_1||V_2|} = (k_1 \cdot A) \cdot \frac{B}{V_2} \overset{=}{\text{or}} k_2 B \cdot \frac{A}{V_1}$$

\# of edges going out of A

prob of hitting B

<u>Pf.</u> Let $\lambda = \sqrt{k_1k_2} > \lambda_1 = \lambda \geq \ldots > \lambda_{n-1} = -\sqrt{k_1k_2}$ be the e.v.'s of A. Let $v_0, \ldots, v_{n-1}$ be orthonormal basis of e. vectors:

$$Av_i = \lambda v_i \text{ and } \langle v_i, v_j \rangle = \delta_{ij}$$

<u>observe</u> if χ_A, χ_B the characteristic function of A and B resp.

$$|E(A,B)| = (\chi_A^t)\, A\, \chi_B$$

$$\begin{pmatrix} 1 & 0 & \overset{A}{1} & 0 & \dots \end{pmatrix} A \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} B =$$

$$= \begin{pmatrix} & & \end{pmatrix} \begin{array}{c} v \\ \vdots \\ v_n \end{array} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{array}{l} \text{if } v \overset{\text{adj}}{\sim} B \quad |E(v,B)| \\ \text{if } v \underset{\text{adj}}{\not\sim} \end{array}$$

$$= \sum_{v \in A} |E(v,B)|$$

$$\chi_A = \sum_{i=0}^{n-1} a_i v_i \ , \ \chi_B = \sum_{i=0}^{n-1} b_i v_i$$

$$|E(A,B)| = \chi_A^t A \chi_B = (\chi_A^t)(\sum b_i \lambda_i v_i) =$$

$$= \sum \lambda_i a_i b_i$$

<u>Recall</u> $v_0 = \cancel{\chi_{V_1} \frac{\sqrt{k_1}}{\sqrt{k_1^2 + k_2^2}} + \chi_{V_2} \frac{\sqrt{k_2}}{\sqrt{k_1^2+k_2^2}}}$ ← normalization of

$$\cancel{\frac{k_1}{k_2} + \frac{k_2}{k_1} = \frac{k_1^2+k_2^2}{k_1 k_2}}$$

$$\tilde{v}_0 = \begin{cases} \frac{1}{\sqrt{k_2}} & v \in V_1 \\ \frac{1}{\sqrt{k_1}} & v \in V_2 \end{cases}$$

$$v_0 = \frac{\tilde{v}_0}{\|v_0\|} = \frac{\hat{v}_0}{\sqrt{V_1 \left(\frac{1}{\sqrt{k_2}}\right)^2 + V_2 \left(\frac{1}{\sqrt{k_1}}\right)^2}} =$$

$$= \frac{\tilde{v}_0}{\sqrt{\frac{V_1}{k_2} + \frac{V_2}{k_1}}} = \frac{\tilde{v}_0}{\sqrt{2\frac{V_1}{k_2}}} = \frac{\tilde{v}_0}{\sqrt{2\frac{V_2}{k_1}}} = \begin{cases} \frac{1}{\sqrt{k_2}} \cdot \frac{\sqrt{k_2}}{\sqrt{2 \cdot |V_1|}} & \text{on } V_1 \\ \frac{1}{\sqrt{k_1}} \cdot \frac{\sqrt{k_1}}{\sqrt{2 \cdot |V_2|}} & \text{on } V_2 \end{cases} =$$

$$= \begin{cases} \frac{1}{\sqrt{2|V_1|}} & \text{on } V_1 \\ \sqrt{2|V_2|} & \text{on } V_2 \end{cases}$$

$$\Rightarrow \quad v_{n-1} = \begin{cases} \frac{1}{\sqrt{2|V_1|}} & \text{on } V_1 \\ -\frac{1}{\sqrt{2|V_2|}} & \text{on } V_2 \end{cases}$$

$$a_0 = \langle \chi_A , u_0 \rangle = |A| \cdot \frac{1}{\sqrt{2|V_1|}}$$

$$b_0 = \langle \chi_B , v_0 \rangle = \frac{|B|}{\sqrt{2|V_2|}}$$

$$a_{n-1} = \chi_A , v_{n-1} \rangle = |A| \cdot \frac{1}{\sqrt{2|V_1|}}$$

$$b_{n-1} = (\chi_B , v_{n-1}) = - \frac{|B|}{\sqrt{2|V_2|}}$$

$$\Rightarrow \quad a_0 b_0 \lambda_0 + a_{n-1} b_{n-1} \lambda_{n-1} = 2 \cdot \sqrt{k_1 k_2} \cdot \cancel{\left(\frac{|A|}{\sqrt{2|V_1|}} + \frac{|B|}{\sqrt{2|V_2|}} \right)} -$$

$$= \cancel{\sqrt{2}\,|A| \sqrt{\frac{k_1 k_2}{|V_1|}} + \sqrt{2}\,|B| \cdot \sqrt{\frac{k_1 k_2}{|V_2|}}}$$

$$= \frac{2|A| \cdot |B| \cdot \sqrt{k_1 k_2}}{\sqrt{2}\,|V_1||V_2|} = \frac{|A||B|}{|V_1||V_2|} \sqrt{k_1 k_2} \Rightarrow$$

$$= \left| |E(A,B)| - \frac{|A||B|}{|V_1||V_2|} \sqrt{k_1 k_2} \right| = \left| \sum_{i=1}^{n-2} a_i b_i \lambda_i \right| \leq$$

Cauchy-Sch

$$\leq \left| \sum_{i=1} |a_i| |b_i| \cdot \lambda \right| \leq \lambda \left| \sum_{i=0}^{n-1} a_i b_i \right| \leq \lambda \sqrt{\|\chi_A\| \cdot \|\chi_B\|} \;\ominus$$

$$\rotatebox{90}{=}\ \langle \chi_A , \chi_B \rangle$$

because $|\lambda_i| \leq \lambda \quad \forall\, i$

$$\ominus \quad \leq \lambda \sqrt{|A| \cdot |B|}$$

since $a_0 b_0 + a_{n-1} b_{n-1} = 0$

☒