05/11/13

Def A simplicial complex (S.C.) is a family of finite subsets X of a set of "vertices" $V = X(0)$, closed under inclusion, i.e. if $F_1 \in X$ and $F_2 \subseteq F_1$ then $F_2 \in X$.

If $|F| = i+1$ say that $\dim(F) = i$, F is i-face, i-simplex, i-cell

$i=0$ — vertices
$i=1$ — edges
$i=2$ — triangles

$\dim X = \sup\{\dim(F) \mid F \in X\}$, $X(i) = \{F \in X \mid \dim F = i\}$

Call X uniform if all max faces of X are of the same dimension.

Max faces = "chambers"
co-dim $=1$ = "walls"

sequence of chambers $c_1, \dots, c_n$ s.t. $\forall i = 1, \dots n-1$: $c_i \cap c_{i+1}$ is a wall

Hypergraph H: is a collection of subsets called "edges". (without requir. of being closed under inclusion)

H d-uniform, if all subset are of size $\underline{d}$.

uniform S.C. $\xrightarrow{\text{uniform}}$ Hypergraph: $H = \{$all max cells of s.c.$\}$

Taking H and make it closed under inclusion. $\leftarrow$

Note A s.c. is completely determined by its maximal faces.

A Coloring of a S.C.

Def Let X be a uniform s.c. of dim d, assume $\exists$ $\tau : X(0) \to \{0, \dots, d\}$ s.t. for every chamber C: $\tau|_C$ is 1-1.

τ is called type function.

If $F \in X(i)$, $F = \{v_0, \dots v_i\}$ then the type of F is the set $\{\tau(v_0), \dots \tau(v_i)\} =: \text{type}(F)$.

If $I \subseteq \{0, \dots, d\}$, then $X(I) = \{F \mid type(F) = I\}$

Def A s.c. X with type function τ is called regular if $I \subset J \subseteq \{0, \dots d\}$, $\exists\, k_{I,J} \in \mathbb{N}$ s.t every face $F \in X(I)$ is contained in exactly $k_{I,J}$ faces $F' \in X(J)$.

Examples $F = \mathbb{F}_q$, $n \in \mathbb{N}$

$\mathcal{A}(n,q)$ = the flag complex of subspaces of $\mathbb{F}_q^n$, i.e. the vertices are all ~~sub~~ proper subspace of $\mathbb{F}_q^n$. ~~st~~ A subset $\{w_0, \dots w_t\}$ form a face if (maybe after re-ordering)

$$\{0\} < W_0 \subset W_1 \subset W_2 \subset \dots \subset W_i \subset \mathbb{F}_q^n$$

This is clearly a uniform s.c.

max face = complete flags:

$$W_1 < W_2 < \dots < W_{n-1} \quad \text{with}$$

~~dim~~ $\dim W_i = i$, i.e.

$$\dim(\mathcal{A}(n,q)) = n-2$$

A type function:

$$\tau(w) = \dim W \longrightarrow \{1, \dots, n-1\}$$

~~This~~ It's regular: $I \subset J$

$I = \{i_1, \dots i_t\}$, $F \in X(I) \Rightarrow$

$\Rightarrow F = \{W_{i_1} < W_{i_2} < \dots < W_{i_t}\}$ $\dim(W_{i_j}) = i_j$

If $G \in X(J)$ then

$$G = \{W_{i_1} < * < W_{i_2} < \dots < W_{i_t}\}$$

of type $\in J$

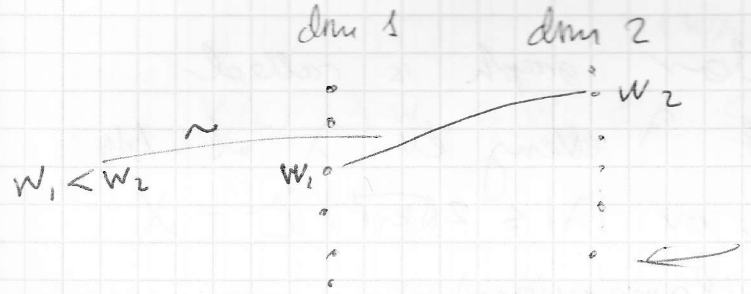
number of ~~st.~~ such G does not depend on F.

Def A uniform s.c. is called chamber complex if every two chambers can be connected by gallery.

Ex Prove that $\mathcal{A}(n,q)$ is a chamber complex.

$n=2$ $\quad \mathcal{A}(n,q)$ is just a set of $(q+1)$ points
($q+1$ - is the number of 1-dim. subspaces of $\mathbb{F}_q^2$)

$n=3$ $\quad$ vertices are all subspaces of dim 1 or 2.



is the same (by duality)

number of the $\dfrac{q^3-1}{q-1}$ ← # of non-zero vectors / ← lying on 1 line

$\begin{bmatrix} n \\ i \end{bmatrix}_q$ = # of subspaces of $\mathbb{F}_q^n$ of dim i

we got a bipartite $(q+1)$-regular graph.

$\mathcal{A}(3,q)$ = the graph of "pts vs lines" of the projective plane over $\mathbb{F}_q$.

$W_2/W_1 \cong \mathbb{F}_q^2$

$$A = \text{adj mat. of } \mathcal{A}(3,q) = \left(\begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array}\right)$$

$$A^2 = \left(\begin{array}{c|c} BB^t & 0 \\ \hline 0 & B^tB \end{array}\right) = \left(\begin{array}{c|c} A^2_{up} & 0 \\ \hline 0 & A^2_{lower} \end{array}\right)$$

Claim

$A^2_{up} = (q+1)I + (J-I) = qI + J$, where

$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, $J = (1)$

$$A^2_{\text{lower}} = (q+1)I + (J - I) \qquad \text{size}\left(A^2_{\text{lower}}\right) = \frac{q^3-1}{q-1} \times \frac{q^3-1}{q-1}$$

$\Downarrow$

e.v of A^2 = twice (e.v. of $qI + J$) $=$

e.v. of J : $\quad q^2+q+1 \quad$ 1 time

$\qquad\qquad 0 \quad q^2+q$ times

$=$ twice ($(q+1)^2$ once , q (q^2+q) times)

So e.v. of A : $\pm(q+1)$, $\pm\sqrt{q}$ (q^2+q) time each

Def A finite connected k-regular graph is called Ramanujan graph if every e.v. λ of Adj satisfies either $|\lambda| = k$, or $|\lambda| \le 2\sqrt{k-1}$

Example $\mathcal{A}(3,q)$ are Ramanujan.

Thm (Alon-Boppana) If $(X_n)_{n=1}^{\infty}$ is a family of connected k-regular graphs with $|X_n| \to \infty$, then

$$\liminf \underbrace{\lambda_1(X_n)}_{2^{nd}\text{ largest e.v.}} \ge 2\sqrt{k-1}.$$

Example $\mathcal{A}(4,q)$ – 2-dimensional s.c.

dim 1 $\qquad$ dim 2 $\qquad$ dim 3

$$\begin{bmatrix} 4 \\ q \end{bmatrix}_1 = \begin{bmatrix} 4 \\ q \end{bmatrix}_3 = \frac{q^4-1}{q-1}$$

$$\begin{bmatrix} 4 \\ q \end{bmatrix}_2 = \frac{(q^4-1)\cdot(q^4-q)}{(q^2-1)\cdot(q^2-q)} =$$

$$=(q^2+1)(q^2+q+1)$$

$k_{1,\{1,3,4\}} = k_{1,3} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = q^2+q+1 = k_{3,1}$

$k_{1,2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = q^2+q+1 = k_{3,2}$

by duality

$k_{2,1} = k_{2,3} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = q+1$

$2(q^3+q^2+q+1) \cdot 2 \cdot (q^2+q+1) =$
$= 2 \cdot (q^2+1)(q^2+q+1) \cdot$
$\cdot (q+1) =$
$= q$

12/11/2015

$\mathbb{F}_q$, $V = \mathbb{F}_q^n$, s.c. $X(0) =$ all non-trivial ($\neq 0, V$) subspaces of V.
$\tau(W) = \dim W$

$X(i) = \{ W_{j_0} \subset W_{j_1} \subset \dots \subset W_{j_i} \}$

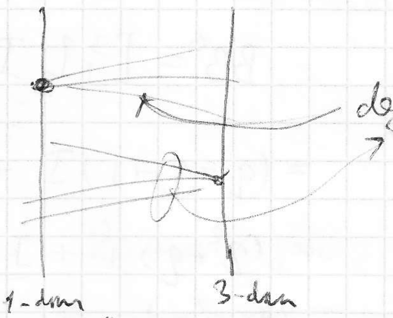
$X = \mathcal{S}(n,q)$.

We've seen last time: $\mathcal{S}(3,q) =$ is a bi-partite $(q+1)$-regular graph = "pts vs lines" of proj. plane.
e.v. $\pm(q+1), \pm\sqrt{q}$.

$\mathcal{S}(4,q)$ – 2-dim-complex, gives rise to three bi-partite graphs:

$B_{1,2}, \; B_{1,3}, \; B_{2,3}$

Let's study $B_{1,3}$ — bi-partite bi-regular graph



degree $= \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = \frac{q^3-1}{q-1}$

$\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q = \frac{q^4-1}{q-1}$ $\qquad \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q$

$A = A_{B_{1,3}} = \left(\begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array} \right)$

adj. mat