$k_{1,\{1,3\}} = k_{1,3} = \begin{bmatrix}3\\1\end{bmatrix}_q = q^2+q+1 = k_{3,1}$

$k_{1,2} = \begin{bmatrix}3\\2\end{bmatrix}_q = q^2+q+1 = k_{3,2}$

by duality

$k_{2,1} = k_{2,3} = \begin{bmatrix}2\\1\end{bmatrix}_q = q+1$

$\cancel{2}(q^3+q^2+q+1) \cdot 2 \cdot \cancel{(q^2+q+1)} =$
$= 2 \cdot (q^2+1)\cancel{(q^2+q+1)} \cdot$
$\cdot (q+1) =$
$= q$

12/11/2015

$\mathbb{F}_q$, $V = \mathbb{F}_q^n$, s.c. $X(0)$ = all non-trivial ($\neq 0, V$) subspaces of V.

$\tau(W) = \dim W$

$X(i) = \{W_{j_0} \subset W_{j_1} \subset \dots \subset W_{j_i}\}$

$X = \mathcal{S}(n,q)$.

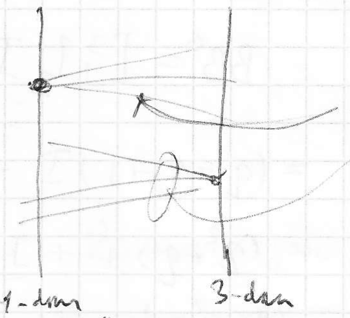
We've seen last time: $\mathcal{S}(3,q) =$ is a bi-partite $(q+1)$-regular graph = "pts vs lines" of proj. plane.

e.v. $\pm(q+1)$, $\pm\sqrt{q}$.

$\mathcal{S}(4,q)$ - 2-dim - complex, gives rise to three bi-partite graphs:

$B_{1,2}$, $B_{1,3}$, $B_{2,3}$

Let's study $B_{1,3}$ — bi-partite bi-regular graph



degree $= \begin{bmatrix}3\\2\end{bmatrix}_q = \frac{q^3-1}{q-1}$

1-dim

3-dim

$\begin{bmatrix}4\\1\end{bmatrix}_q = \frac{q^4-1}{q-1}$

$\begin{bmatrix}4\\3\end{bmatrix}_q = \begin{bmatrix}4\\1\end{bmatrix}_q$

$A = A_{B_{1,3}} = \left(\begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array}\right)$

adj. mat.

$$A^2 = \begin{pmatrix} BB^t & 0 \\ 0 & B^tB \end{pmatrix}$$

$$BB^t = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q I + (q+1)(J - I) \;=$$

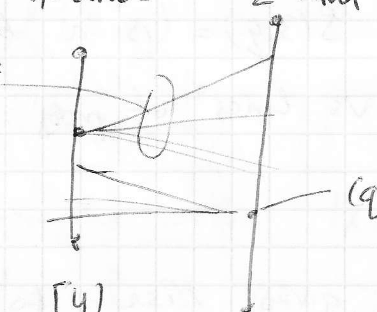
there are $(q+1)$ 3-dim spaces containing this 2-dim space

If w_1, w_1' : unique 2-dim space containing both of them

$$= (q^2+q+1) I + (q+1)(J - I) = q^2 I + (q+1) J$$

$$e.v(A) = \underbrace{\pm q}_{\text{many times}}, \; \pm\sqrt{(q^2+q+1)^2} = \pm (q^2+q+1)$$

$B_{1,2}$



$$|E| = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{(q^4-1)(q^4-q)}{(q^2-1)(q^2-q)} = (q^2+1)(q^2+q+1)$$

$$A_{adj} = \left(\begin{array}{c|c} 0 & B \\ \hline B^t & 0 \end{array}\right)$$

$$e.v(BB^t) \cup \{0\} = e.v.(B^tB) \cup \{0\}$$

1-dim $\to$ 2-dim $\to$ 1-dim

$$BB^t = \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q I + (J - I) =$$

$$= (q^2+q+1) I + (J - I) =$$

$$= (q^2+q) I + J$$

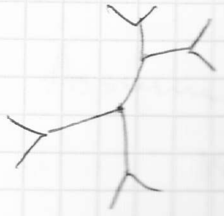
$$e.v.(BB^t) = \underbrace{q^2+q}_{\text{many times}}, \; \cancel{(deg)}^2 \text{ degree}(\to) \cdot \text{degree}(\leftarrow)$$

$$e.v.(A_{adj}) = \pm\sqrt{q^2+q}, \; \pm\sqrt{\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q}$$

$$\| \\ \pm\sqrt{(q^2+q+1)(q+1)}$$

Recall A finite conn. k-reg. graph X is called Ramanujan, if every e.v. of it: either $\lambda = \pm k$
or $|\lambda| \le 2\sqrt{k-1}$

Such X is covered by the k-regular tree T_k.



$$X = \frac{T_k}{\Gamma}, \text{ where } \Gamma = \pi_1(X) \text{ acting on } T_k$$

$$A : L^2(T_k) \to L^2(T_k)$$

$$\mathrm{Spec}(A) \underset{\uparrow}{=} [-2\sqrt{k-1}, 2\sqrt{k-1}]$$

Exercise

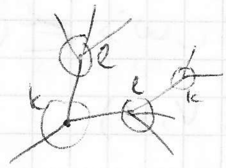
Def If X is a finite graph, T - its universal cover (a tree), X is called Ramanuj. if all e.v. of X are either trivial or in $\mathrm{Spec}(A_T)$

Alon-Boppana (Greenberg (see his thesis in Hebrew)): This is the best bound you can hope for an infinite family covered by T.

Two diff. Ramanujans: every e.v. downstairs is upstairs
Is the $\mathrm{Spec}(A_T)$ continuous path.
Or every e.v. downstairs either trivial or $\le \|A_T\|$.

Now $T = T_{k,l}$, then $\|A_{T_{k,l}}\| = \sqrt{k-1} + \sqrt{l-1}$

$T_{k,l}$:

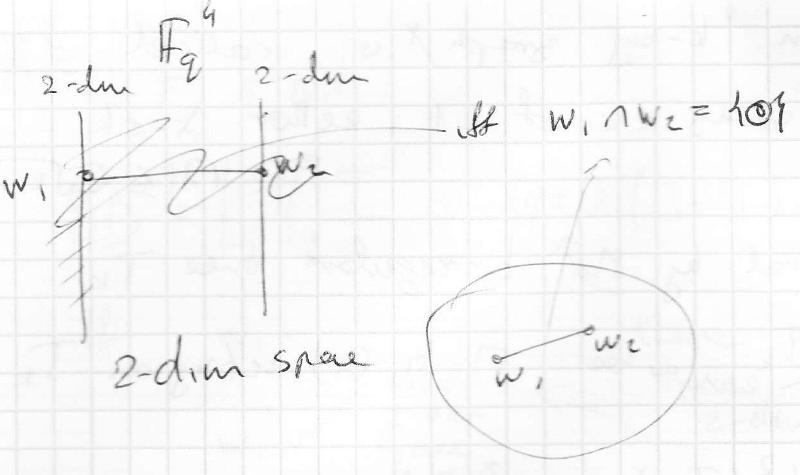


Therefore $B_{1,2}$ is Ramanujan, hence

$$\sqrt{q^2+q} \le \sqrt{q+1-1} + \sqrt{q^2+q+1-1} = \sqrt{q} + \sqrt{q^2+q}$$

Exercise: Fix n, B_{ij} = "i-dim vs j-dim" in $\mathbb{F}_q^n$ all Ramanujan

Hint:



$$\{0\} = V_0 < V_1 < \dots < V_n \leq \mathbb{F}_q^n$$

Every 2 chambers V, U are connected by a gallery

$$D_i(V,U) := \dim(V_i \cap U_i)$$

$$D(V,U) = \sum D_i(V,U)$$

$$d(V,U) = \left[\sum_{i=0}^{n-1} i\right] - D(V,U)$$

<u>Prop</u> If $d(V,U) > 0$, then there are 2 chambers V', U' s.t. $|V' \cap V| \geq n$ or $|U' \cap U| \geq n$

$$d(V', U') < d(V, U)$$

i.e. or $U = U'$ or they intersect by wall.

<u>Proof</u> $\exists i$: $D_i(V,U) = D_{i+1}(V,U) <$
$< D_{i+2}(V,U)$

pick $w \in (V_{i+2} \cap U_{i+1}) \setminus (V_i \cap U_i)$ a vector

replace: V_{i+1} with $\langle V_i, w\rangle \in V'_{i+1}$
U_{i+1} with $\langle U_i, w\rangle \in U'_{i+1}$

then $D_{i+1}(V', U') > D_{i+1}(V, U)$ $\boxtimes$

Size of the gallery G connecting U and V

$$d(U,V) \leq |G| \leq 2d(U,V)$$

and $d(U,V) = \sum i - D(U,V)$.

Another proof. Induction on n.

$n=3$: $\quad W_1 \leq W_2 \rightsquigarrow U_1 \leq U_2$
$\quad$ dim: 1 $\quad$ 2

$\exists$ gallery: $\{W_1 \leq W_2\}$, $\{W_2 \cap U_2 \leq W_2\}$,
$\{W_2 \cap U_2 \leq U_2\}$, $\{U_1 \leq U_2\}$.

Assume that $\mathcal{S}(n+1,q)$ is a chamber complex.

$W_*^i = \{W_1^i \leq W_2^i \leq \ldots \leq W_n^i\}$ $\; i=1,2$ are two chambers in $\mathcal{S}(n+1,q)$

Denote $\quad W_{n-1}^3 = W_n^1 \cap W_n^2$

let $\quad W_*^3 = \{W_1^3 \leq \ldots \leq W_{n-1}^3\}$ be a chamber in $\mathcal{S}(n,q)$

where $W_1^3, \ldots, W_{n-2}^3$ arbitrary, $W_{n-1}^3 = W_n^1 \cap W_n^2$

Identify W_n^1 with $\mathbb{F}_q^n$, we can think of
$\widetilde{W}_*^1 = \{W_1 \leq \ldots \leq W_{n-1}\}$ as
a chamber in $\mathcal{S}(n,q)$

We have a gallery $C_0, \ldots C_r$ in $\mathcal{S}(n,q)$ with
$C_0 = \widetilde{W}_*^1$ and $C_r = W_*^3$

We identify W_n^2 with $\mathbb{F}_q^n$, build a gallery in
$\mathcal{S}(n,q)$ $\; C_0', \ldots C_s'$ with $C_0' = W_*^3$, $\; C_s' = \widetilde{W}_*$

Construct a gallery in $\mathcal{S}(n+1,q)$ by
$C_0 \cup \{W_n^1\}, \ldots C_r \cup \{W_n^1\}, C_0' \cup \{W_n^2\} \ldots C_s' \cup \{W_n^2\}$

If d_n is a diam of $\mathcal{S}(n,q)$, then by a
smart choice of W_3, $d_{n+1} = n + d_n$.

19/11/13

Assume X is a simp complex on a finite set
of vertices $V = X(0)$. Assume X is pure, i.e.
all max faces are of the same size $d+1$, in part, $\dim X = d$.

Assume $\tau: V \to [0, \ldots, d]$ s.t. $\forall$ max face F,
$\tau|_F$ is 1-to-1 & onto