Another proof: Induction on n.

$n=3$: $W_1 \subseteq W_2 \rightsquigarrow U_1 \subseteq U_2$

dim: 1, 2

A gallery: $\{W_1 \subseteq W_2\}$, $\{W_2 \cap U_2 \subseteq W_2\}$,

$\{W_2 \cap U_2 \subseteq U_2\}$, $\{U_1 \subseteq U_2\}$.

Assume that $S(n+1,q)$ is a chamber complex

$W_*^i = \{W_1^i \subseteq W_2^i \subseteq \dots \subseteq W_n^i\}$ $i=1,2$ are two chambers in $S(n+1,q)$

Denote $W_{n-1}^3 = W_n^1 \cap W_n^2$

let $W_*^3 = \{W_1^3 \subseteq \dots \subseteq W_{n-1}^3\}$ be a chamber in $S(n,q)$

where $W_1^3, \dots, W_{n-2}^3$ arbitrary, $W_{n-1}^3 = W_n^1 \cap W_n^2$

Identify W_n^1 with $\mathbb{F}_q^n$, we can think of $\tilde{W}_*^1 = \{W_1 \subseteq \dots \subseteq W_{n-1}\}$ as a chamber in $S(n,q)$

We have a gallery $C_0, \dots C_r$ in $S(n,q)$ with $C_0 = \tilde{W}_*^1$ and $C_r = W_*^3$

We identify W_n^2 with $\mathbb{F}_q^n$, build a gallery in $S(n,q)$ $C_0', \dots C_s'$ with $C_0' = W_*^3$, $C_s' = \tilde{W}_*$

Construct a gallery in $S(n+1,q)$ by

$C_0 \cup \{W_n^1\}, \dots C_r \cup \{W_n^1\}, C_0' \cup \{W_n^2\} \dots C_s' \cup \{W_n^2\}$

If d_n is a diam of $S(n,q)$, then by a smart choice of W_3, $d_{n+1} = n + d_n$.

19/11/13

Assume X is a simp complex on a finite set of vertices $V = X(0)$. Assume X is pure, i.e. all max faces are of the same size $d+1$, in part, $\dim X = d$.

Assume $\tau: V \to [0, \dots, d]$ s.t. $\forall$ max face F, $\tau|_F$ is 1-to-1 & onto

Assume (X,τ) is <u>regular</u> if for every $I \subset J \subseteq [0,\dots,d]$ for every F of type I (i.e. $\tau(F) = I$) the number of faces of type J containing it, is $k_{I/J}$, i.e. depends only on I and J but not on F.

<u>Exercise</u> $\mathcal{B}(n,q)$ are regular w.r.t. this definition.

<u>Notation</u> Denote $V_i = \tau^{-1}(i) \leftarrow$ all the vertices of type i.

<u>Def</u> Let $A_{i_1}, \dots A_{i_t}$, $A_{i_j} \subseteq V_{i_j}$, where $I = \{i_1, \dots, i_t\} \subseteq \{0, \dots, d\}$, denote

$E(A_{i_1}, \dots, A_{i_t})$ = the set of all faces of type I, whose vertices are in $\bigcup A_{i_j}$.

Clearly, exactly one vertex in each A_{i_j}.

<u>denote</u> $X(I)$ = all faces of type I

define $\operatorname{disc}_{(X,\tau)}(A_{i_1}, \dots A_{i_t}) =$

$$= \left| \frac{|E(A_{i_1}, \dots A_{i_t})|}{|X(I)|} - \prod_{j=1}^{t} \frac{|A_{i_j}|}{|V_{i_j}|} \right|$$

If $I = \{0, \dots, d\}$

$$\operatorname{Disc}(X,\tau) = \max_{\substack{\{A_0, A_1, \dots A_d \\ A_i \subset V_i\}}} \operatorname{disc}_{(X,\tau)}(A_0, \dots, A_d)$$

<u>Recall</u> If B is bi-partite (k_1, k_2) - bi-regular graph. It is a regular pure simp complex.

Mixing lemma:

$$\left| \frac{|E(A,B)|}{|E|} - \frac{|A|\cdot|B|}{|V_1|\cdot|V_2|} \right| \leq \frac{\lambda(B)}{\sqrt{k_1 k_2}} \cdot \sqrt{\frac{|A|}{|V_1|}\frac{|B|}{|V_2|}} \leq$$

$$\leq \frac{\lambda(B)}{\sqrt{k_1 k_2}}$$

$\lambda^{nor}(B) := \frac{\lambda(B)}{\sqrt{k_1 k_2}}$ (is it $\lambda(B)$ of the normal laplacian ?)

i.e. mixing lemma $\Rightarrow$ $Disc_{(B,\tau)} \leq \lambda^{nor}(B)$

For I and J, $I \cap J = \emptyset$, $I, J \subseteq \{0, \dots d\}$

Define $B_{I,J} = (X(I), X(J), E)$

$F_1 \in X(I)$ and $F_2 \in X(J)$ are connected iff $F_1 \cup F_2 \in X(I \cup J)$

A special case: $B_i := B_{\{i\}, \{0, \dots, d\} \setminus \{i\}}$

Proposition (X, τ) as before

$$Disc_{(X,\tau)} \leq \sum \lambda^{nor}(B_i) \leq (d+1) \max_i \lambda^{nor}(B_i)$$

Lemma 0 $A_0 = V_0$, $A_1, \dots A_d$, $A_i \subseteq A_i$, $i > 0$

$$disc_{(X,\tau)}(V_0, A_1, \dots A_d) = disc_{(X,\tau)}(A_1, \dots A_d)$$

Pf

$$\left| \frac{|E(V_0, A_1, \dots A_d)|}{|E(V_0, V_1, \dots V_d)|} - \prod_{i=0}^{d} \frac{|A_i|}{|V_i|} \right| \;=$$

$[d] = \{0, \dots, d\}$

$$\left| \frac{|E(A_1, \dots A_d)|}{|E(V_1, \dots V_d)|} \right.$$

$$= \left| \frac{k_{[d]\setminus\{0\}, [d]} \, |E(A_1, \dots, A_d)|}{k_{[d]\setminus\{0\}, [d]} \, |E(V_1, \dots V_d)|} - \prod_{i=1}^{d} \frac{|A_i|}{|V_i|} \right| = disc_{X,\tau}(A_1, \dots A_d)$$

since $A_0 = V_0$ ∎

Lemma 1 $disc_{(X,\tau)}(A_0, A_1, \dots A_d) \leq disc_{B_0}(A_0, E(A_1, \dots A_d)) +$

$$+ \frac{|A_0|}{|V_0|} \cdot disc_X(A_1, \dots A_d).$$

Pf. $\left| \dfrac{|E(A_0,\dots A_d)|}{|E(V_0,\dots,V_d)|} - \prod_{i=0}^{d} \dfrac{|A_i|}{|V_i|} \right| =$

$$= \left| \frac{|E_{B_0}(A_0, \overbrace{E(A_1,\dots A_d)}^{E'})|}{|E(V_0,\dots V_d)|} - \frac{|A_0|}{|V_0|}\cdot\frac{|E'|}{|E([d]\setminus\{0\})|} + \right.$$

$$\left. + \frac{|A_0|}{|V_0|}\left(\frac{|E'|}{|E([d]\setminus\{0\})|} - \prod_{i=1}^{d}\frac{|A_i|}{|V_i|}\right)\right| \le$$

$$\le \operatorname{disc}_{B_0}(A_0, E') + \frac{|A_0|}{|V_0|}\cdot \operatorname{disc}_X(A_1,\dots A_d)$$

triangle ineq. ☒

Finally $\operatorname{disc}_{(X,\tau)}(A_0,\dots A_d) \le$

$$\le \operatorname{disc}_{B_0}(A_0, E(A_1,\dots A_d)) + \frac{|A_0|}{|V_0|}\operatorname{dis}_X(A_1,\dots,A_d)$$

$$\le \lambda^{nor}(B_0) + \underbrace{\frac{|A_0|}{|V_0|}}_{\le 1}\cdot \operatorname{disc}_{X,\tau}(V_0, A_1,\dots A_d) \le$$

Lemma 0

$$\le \lambda^{nor}(B_0) + \lambda^{nor}(B_1) + \operatorname{dis}_{X,\tau}(V_0, V_1, A_2,\dots A_d) \le$$

$$\le \dots \le \sum \lambda^{nor}(B_i) + \underbrace{\operatorname{disc}(V_0, V_1, \dots V_d)}_{=0}$$

☒

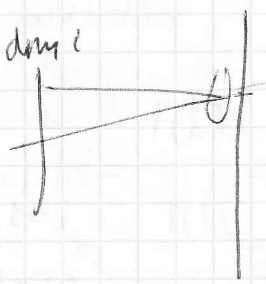
Exercise (Challenging) $X = S(n,q)$ ~ evaluate $\lambda(B_i)$:

walls at level i:

dim i

B_i

bi-partite bi-reg graph



$\deg = q+1$

Fix n, $q \to \infty$

$\max_i \lambda^{nor}(B_i) \to 0$.