Uriya — The affine building associated to $GL_n(F)$ — 26/11

Valuation Rings

Def F - field. An (additive) discrete valuation on F is a map

$$v : F^* \xrightarrow[\text{onto}]{} \mathbb{Z}$$

s.t.
1) $v(ab) = v(a) + v(b)$
2) $v(a+b) \geq \min\{v(a), v(b)\}$

Set $v(0) = \infty$

Define $\mathcal{O}$ valuation ring to be

$$\mathcal{O} = \{x \in F \mid v(x) \geq 0\}$$

Claims
1) $\mathcal{O}$ is a ring
2) $\mathcal{O}^* = \{x \in F \mid v(x) = 0\}$
3) Let $\pi \in F$ s.t. $v(\pi) = 1$, then the ideals of $\mathcal{O}$ are

$$\mathcal{O}, \pi\mathcal{O}, \pi^2\mathcal{O}, \dots$$

Proof 1) $1 \in \mathcal{O}$ since $v(1) = v(1 \cdot 1) = v(1) + v(1)$
$\Rightarrow v(1) = 0$

likewise, $(-1) \in \mathcal{O}$,

$a, b \in \mathcal{O} \Rightarrow v(ab) = v(a) + v(b) \geq 0 \Rightarrow$
$\Rightarrow ab \in \mathcal{O}$.

likewise, $a, b \in \mathcal{O} \Rightarrow a+b \in \mathcal{O}$

2) If $v(x) = 0$, then $v(x^{-1}) = -v(x) = 0$
$\Rightarrow x^{-1} \in \mathcal{O} \Rightarrow x \in \mathcal{O}^*$

If $x \in \mathcal{O}^*$, let $y = x^{-1} \in \mathcal{O}$, then
$v(x), v(y) \geq 0$, but $\underset{\overset{\parallel}{v(1) = 0}}{v(xy)} = v(x) + v(y) \Rightarrow$
$\Rightarrow v(x) = v(y) = 0$

3) Let $0 \neq I \triangleleft \mathcal{O}$, take $x \in I$ s.t. $v(x)$ is minimal

$\hookrightarrow$

let $n = v(x)$, then

$$v(\pi^{-n} x) = v(\pi)\cdot(-n) + v(x) = 0$$

$\Rightarrow \pi^{-n}\cdot x \in \mathcal{O}^* \Rightarrow$

$$\Rightarrow \pi^n = x \cdot \underbrace{(\pi^{-n}\cdot x)^{-1}}_{\mathcal{O}^*} \in I \Rightarrow \pi^n \mathcal{O} \subseteq I$$

On the other hand, if $y \in I$, then

$v(y) \geq v(x) = n \Rightarrow v(\pi^{-n} y) = -n + v(y) \geq 0$

$\Rightarrow \pi^{-n} y \in \mathcal{O} \Rightarrow y \in \pi^n \mathcal{O}$, i.e.

$I = \pi^n \mathcal{O}$ $\boxed{\times\!\!\times}$

<u>Claim</u> fix $0 < \varepsilon < 1$, set $|x|_v = \varepsilon^{v(x)}$

and $d(x,y) = |x-y|_v$

then F becomes a metric space with respect to d

<u>Proof</u>: exercise

<u>Remark</u> $|\ |_v : F^* \to \mathbb{R}_{>0}$ satisfies

① $|xy|_v = |x|_v |y|_v$

② $|x+y|_v \leq \max\{|x|_v, |y|_v\}$

set $|0|_v = 0$.

$|\cdot|_v$ with these properties is called a mult. valuation

<u>Example</u> ① $F = \mathbb{Q}$, p-prime number

$v_p : \mathbb{Q}_p^* \to \mathbb{Z}$

$v_p(\frac{a}{b} p^m) = m$

$\pi = p$ satisfies $v_p(p) = 1$

$$\mathcal{O}_{v_p} = \left\{ \frac{m}{n} \,\middle|\, \begin{matrix} m,n \in \mathbb{Z} \\ (n,p)=1 \end{matrix} \right\} = \mathbb{Z}\left[\frac{1}{q} \,\middle|\, \begin{matrix} q\text{-prime} \\ p \neq q \end{matrix} \right]$$

The completion of $\mathbb{Q}$ at v_p is

$\mathbb{Q}_p$ - the p-adic numbers.

2) $F = \mathbb{F}_q(t)$

$v : F^* \to \mathbb{Z}$

$v\left(\frac{f(t)}{g(t)}\right) =$ "order of zero/pole at $t=0$"

$v\left(\frac{\tilde{f}}{\tilde{g}} \cdot t^m\right) = m$

$\tilde{f}(0) \neq 0 \neq \tilde{g}(t)$

$\pi = t$ satisfies $v(t) = 1$

$$\mathcal{O}_v = \left\{ \frac{f(t)}{g(t)} \;\middle|\; \begin{array}{c} f(t), g(t) \in \mathbb{F}_p[t] \\ g(0) \neq 0 \end{array} \right\}$$

<u>Remark</u> We can complete F w.r.t. the metric d. The resulting metric space F_v is again a field and

v extends to F_v by

$$v(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} v(x_n)$$

The completion of $\mathbb{F}_p(t)$ at v is $\mathbb{F}_p((t)) = \left\{ \sum_{i=m}^{\infty} a_i t^i \;\middle|\; \begin{array}{c} a_i \in \mathbb{F}_p \\ m \in \mathbb{Z} \end{array} \right\}$

Another descriptions of $\mathbb{Q}_p$, $\mathbb{Z}_p$ - val ring of $\mathbb{Q}_p$ (p-adic integers)

$$\mathbb{Q}_p = \left\{ a_m p^m + a_{m+1} p^{m+1} + \dots \;\middle|\; \begin{array}{c} a_i \in \{0, \dots, p-1\} \\ m \in \mathbb{Z} \end{array} \right\} =$$

$$= \varprojlim \left(\dots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \right)$$

Henceforth F - field, $v : F^* \to \mathbb{Z}$ discrete val.

$\mathcal{O} = \mathcal{O}_v = \{x \in F \mid v(x) \geq 0\}$, π - uniformizer ($v(\pi) = 1$)

$k = \mathcal{O}/\pi\mathcal{O}$ - residue field,

$2 \le d$ - fixed integer

Lattices

Dfn A $\mathcal{O}$-lattice in F^d is a f.g. $\mathcal{O}$-module $L \subseteq F^d$ s.t. L contains a basis of F^d (alternatively, $F \cdot L = F^d$)

Example $L = \mathcal{O}^d \subseteq F^d$

$$\mathcal{O}^d = \mathcal{O}e_1 + \mathcal{O}e_2 + \dots + \mathcal{O}e_d$$

Example Let $v_1, \dots, v_d$ be an F-basis of F^d. Then $L = \mathcal{O}v_1 + \mathcal{O}v_2 + \dots + \mathcal{O}v_d$ is a lattice

Fact All lattices are of this form.

If $L = \mathcal{O}v_1 + \dots + \mathcal{O}v_d$ and $\{v_1, \dots, v_d\}$ is a basis of F^d then we say that $\{v_1, \dots, v_d\}$ is a basis of L.

Lemma If $L_1 \subset L_2$ lattices in F^d, then there exists a basis $v_1, \dots, v_d$ of L_2 and integers $n_1, \dots, n_d$ s.t.

$$L_1 = \mathcal{O}\pi^{n_1}v_1 + \dots + \mathcal{O}\pi^{n_d}v_d.$$

Proof (Sketch) Choose $v_1, \dots, v_d$ of L_2, $u_1, \dots, u_d$ of L_1

write the transition matrix.

$$u_i = \sum_j \alpha_{ij} v_j \qquad \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1d} \\ & & \\ \alpha_{d1} & & \alpha_{dd} \end{pmatrix} \in M_d(\mathcal{O})$$

$$\downarrow$$

$$\begin{bmatrix} & & \\ & n_{ij} & \\ & & \end{bmatrix} \text{ valuation matrix}$$

$$\downarrow$$

smallest valuation

$$\begin{pmatrix} \boxed{\ } & 0 & 0 & 0 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{pmatrix}$$

$\downarrow$ induction

...

$$\frac{\frac{3}{2}k_0 \cdot 3k_1 - k_0 k_1}{\frac{3}{2}k_0 \cdot 3k_1} = $$

$$= \frac{7}{9}$$

$\frac{3}{a}$

Actions of GLd and $PGLd$

$GLd(F) = \{$ inv $d \times d$ matr. over $F\}$

~~PGL~~ $(\mathcal{O})$ $/\mathcal{O})$

$GLd(F)$ acts transitively on lattices

$$gL = g\left(\sum \mathcal{O} v_i\right) = \sum \mathcal{O} g v_i$$

$GL_d(F)$ acts transitively since it acts transitively on bases of F^d

<u>Dfn</u> $L, L' \underset{lat}{\leq} F^d$ are equiv $(L \sim L')$ if $\exists\ x \in F^*$ s.t. $L = xL'$.

$[L]$ - equiv class of L

$GL_d(F)$ acts on $\{[L] \mid L \underset{lat}{\leq} F^d\}$

$\Rightarrow$ $PGL_d(F) = GL_d(F)/_{F^*}$

acts $\{[L] / L \underset{lat}{\leq} F^d$

<u>Notice</u>

$$\mathrm{Stab}_{GLd(F)} \mathcal{O}^d = GLd(\mathcal{O})$$

$$\mathrm{Stab}_{PGLd(F)} [\mathcal{O}^d] = PGLd(\mathcal{O}) = GLd(\mathcal{O})/_{\mathcal{O}^*}$$

Construction of the building

<u>Conclusion</u> lattices $\cong GLd(F)/_{GLd(\mathcal{O})}$

$$g\mathcal{O}^d \longleftrightarrow g\, GLd(\mathcal{O})$$

$$v_1\mathcal{O} + \dots + v_d\mathcal{O} \longrightarrow [v_1 \dots v_d]\, GLd(\mathcal{O})$$

equiv classes of lat $\cong PGLd(F)/_{PGLd(\mathcal{O})}$

$$[g\mathcal{O}^d] \Longleftrightarrow g \cdot PGLd(\mathcal{O})$$

The affine building of $GL_d(F)$

We construct an infinite simplicial complex X as follows:

Vertices $X(0) = \{ [L] \mid L \underset{\text{lat}}{\leq} F^d \}$

Edges $[L]$ is adj to $[L']$ if $\exists\, x$ s.t.

$$\pi L \leq x L' \leq L$$

The relation is symmetric since

$$\pi \cdot L' \leq x^{-1} \pi L \leq L'$$

High cells i-dim cells $= \{ [L_1], \dots, [L_{i+1}] \}$ distincts s.t.

$[L_a] \underset{\text{adj}}{\sim} [L_b]$ for all a, b.

Claim X is $(d-1)$-dimensional, more precisely:

① every cell is contained in $(d-1)$-cell

② there are no d-cells, i.e. $X(d) = \emptyset$.

Proof ① Fix some i-cell

$\{ [L_1], \dots, [L_{i+1}] \}$ w.l.o.g. $L_{i+1} = \mathcal{O}^d$

(otherwise mult by suitable element of $PGL_d(F)$.)

$\{ [L_1], \dots, [L_i] \}$ adj to $\mathcal{O}^d$

$$\Rightarrow \pi \mathcal{O}^d \leq \begin{matrix} L_1 \\ \vdots \\ L_i \end{matrix} \leq \mathcal{O}^d$$

Moreover since $[L_a] \sim [L_b] \Rightarrow L_a \leq L_b$ or $L_b \leq L_a$

w.l.o.g.

$$\pi \mathcal{O}^d = \pi L_{i+1} \lneq L_1 \lneq L_2 \lneq \dots \lneq L_i \leq \mathcal{O}^d$$

mod by $\pi \mathcal{O}^d$

$$0 \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_i \leq k^d \quad , \quad k = \mathcal{O}/\pi\mathcal{O}$$

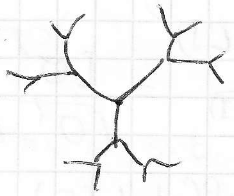
k-vector spaces.

The longest chain we can have is of length $d-1 \Rightarrow i \leq d-1$

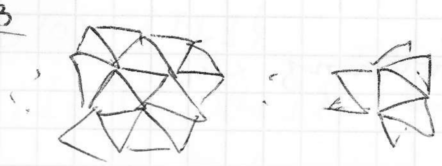
Also the chain $\{V_n\}_{n=1}^{i}$ can be refined to a chain of length $d-1$, which in turn gives rise to a chain of $(d-1)$ lattices, i.e. $(d-1)$-cell containing our i-cell $\boxtimes$

__Fact__ X is connected and contractible.

__In fact__ if $d=2$, $|k|=q$, then X is the $(q+1)$-regular tree



__$d=3$__



__Coloring__ $c: X \xrightarrow[\text{color function}]{} \mathbb{Z}/d$

$$c(g\,PGL_d(\mathcal{O})) = v(\det g) \mod d.$$

equiv.

$$c([L]) = \log_{|k|}[\mathcal{O}^d : L] \mod d$$

choose $L \leq \mathcal{O}^d$

__Example__ $c(\mathcal{O}^d) = 0$

$$c(\mathcal{O}\pi e_1 + \mathcal{O}e_2 + \dots + \mathcal{O}e_d) = 1$$

__Remark__ The action of $PGL_d(F)$ does not preserve colors!

However the action of $SL_d(F)$ does preserve colors.

If C is a cell then

$$c(C) = \{ c([L]) \mid [L] \in C \}$$

$$|c(C)| = \dim C + 1 = |C|$$

Def If (x, y) is a directed edge then

$$c(x,y) := c(y) - c(x)$$

Note $PGL_d(F)$ preserves the color of directed edges

Proof $x = [L]$, $y = [L']$

$$L \leq L' \leq \mathcal{O}^d$$

$$q^{c(x,y)} = q^{c([L']) - c([L])} =$$

$$= \frac{q^{c([L'])}}{q^{c([L])}} = \frac{[\mathcal{O}^d : L']}{[\mathcal{O}^d : L]} = [L' : L]$$

but $\forall g \in PGL_d(F)$ $[gL' : gL] = [L' : L]$ ☒

Prp X is regular in the following sense

If $\emptyset \neq I \subseteq J \subseteq \mathbb{Z}/d$

and $C \in X$ is a cell of color I, then

$$\# \{ C' \mid \begin{array}{l} c(C') = J \\ C' \supseteq C \end{array} \}$$ is ind. of C.

Thm $PSL_d(F)$ acts transitively on $(d-1)$-cells.

13.12.12 Alex

F - local field, non-arch (e.g. $F = \mathbb{Q}_p$ or $\mathbb{F}_q((x))$)

$\mathcal{O}$ = ring of integers in F e.g. ($\mathcal{O} = \mathbb{Z}_p$ or $\mathbb{F}_q[[x]]$)

$$\mathcal{O} = \{ x \in F \mid val(x) \geq 0 \}$$