However the action of $SL_d(F)$ does preserve colors.

If C is a cell then

$$c(C) = \{ c([L]) \mid [L] \in C \}$$

$$|c(C)| = \dim C + 1 = |C|$$

Def If (x,y) is a directed edge then

$$c(x,y) := c(y) - c(x)$$

Note $PGL_d(F)$ preserves the color of directed edges

Proof $x = [L]$, $y = [L']$

$L \leq L' \leq \mathcal{O}^d$

$$q^{c(x,y)} = q^{c([L']) - c([L])} =$$

$$= \frac{q^{c([L'])}}{q^{c([L])}} = \frac{[\mathcal{O}^d : L']}{[\mathcal{O}^d : L]} = [L' : L]$$

but $\forall g \in PGL_d(F)$ $[gL' : gL] = [L' : L]$ ☒

Prp X is regular in the following sense

If $\emptyset \neq I \subseteq J \subseteq \mathbb{Z}/d$

and $C \in X$ is a cell of color I, then

$$\# \{ C' \mid \substack{c(C') = J \\ C' \supseteq C} \} \text{ is ind. of } C.$$

Thm $PSL_d(F)$ acts transitively on $(d-1)$-cells.

3.12.12 Alex

F - local field, non-arch (e.g. $F = \mathbb{Q}_p$ or $\mathbb{F}_q((x))$)

$\mathcal{O}$ = ring of integers in F e.s. ($\mathcal{O} = \mathbb{Z}_p$ or $\mathbb{F}_q[[x]]$)

$$\mathcal{O} = \{ x \in F \mid val(x) \geq 0 \}$$

m = the max ideal of $\mathcal{O} = \{x \in \mathcal{O} \mid val(x) > 0\} = (\pi)$

(e.g. $\pi = p$ for $\mathbb{Z}_p$, or $\pi = x$)

$V = F^d$, let L be a f.g. $\mathcal{O}$-submodule of V, which generates V over F

<u>Exercise</u> $\exists$ a basis $\alpha_1, \dots \alpha_d$ of V over F s.t.

$L = \mathcal{O}\alpha_1 + \dots + \mathcal{O}\alpha_d$

$L_1 \sim L_2$ if $\exists\ 0 \neq t \in F$ s.t. $tL_1 = L_2$ (equiv. rel)

$[L]$ - equiv class

<u>Every</u> $0 \neq t \in F$ can be written uniquely as

$t = \pi^n \lambda$, $n \in \mathbb{Z}$, $\lambda \in \mathcal{O}^*$, $n = val(t)$

$\mathcal{O}^* = \{x \in \mathcal{O} \mid x \text{ is invertible in } \mathcal{O}\} \overset{\text{Exer.}}{=} \mathcal{O} \setminus m =$

$= \{x \in \mathcal{O} \mid val(x) = 0\}$.

$\mathcal{O}/m = \mathbb{F}_q$ the residue field.

If $L_1 \sim L_2$, then $\exists\, n \in \mathbb{Z}$ s.t. $L_1 = \pi^n L_2$

Building

$\mathcal{B}d(F)$ = simplicial complex on the set of vertices

$\mathcal{B}(0) = \{[L]\}$

$\{[L_0], \dots, [L_i]\}$ form an i-cell if $\exists\ L_i' \in [L_i]$ s.t.

$\pi L_i' \subset L_0' \subset L_1' \subset \dots \subset L_i'$

<u>Note</u> $L/_{\pi L} \simeq \mathbb{F}_q^d$

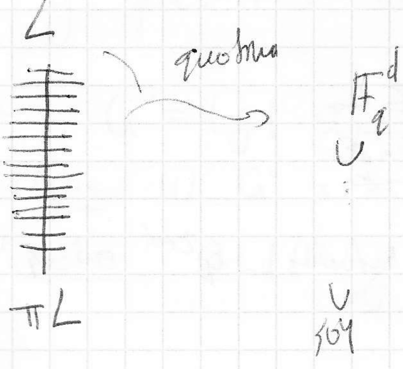
$\alpha = \frac{k}{n-1}$

$2 = x_\alpha \leq d+1 -$

$= 1 + \cdot$??

$\dim \mathcal{B}d(F) \leq d-1$, In fact, $\dim \mathcal{B}d(F) = d-1$, since we can present a $(d-1)$-cell:

Moreover, $\mathcal{B}d(F)$ is pure. Any cell can be made "finer"...



$[L_1] \underset{edge}{—} [L_2] \iff \exists L_1' \in [L_1], L_2' \in [L_2]$

$$\pi L_1' \subset \pi L_2' \subset L_1' \subset L_2'$$

Remarks: if $\{[L_0], \ldots, [L_i]\}$ is an i-cell, then clearly $[L_i] - [L_j]$

Exercise: if $\{[L_0], \ldots [L_i]\}$ s.t. $[L_t] - [L_s]$

$\forall\ 0 \le t \ne s \le i$, then $\{[L_0], \ldots, [L_i]\}$ is an i-cell

type function / coloring

$$\tau : B(0) \to \mathbb{Z}/d\mathbb{Z}$$

Let $L_0 = \mathcal{O}e_1 + \cdots + \mathcal{O}e_d$, where $\{e_1, \ldots, e_d\}$ the standard basis

given L, $\exists$ a ~~unique~~ $n \in \mathbb{Z}$ s.t. $\pi^n L \subseteq L_0$

define $\tau([L]) = \log_q (L_0 : \pi^n L)$ mod d

Claim (1) τ is well-defined

(2) $L = \alpha_1 \mathcal{O} + \cdots \mathcal{O}\alpha_d$

$M = (\alpha_1 | \alpha_2 | \cdots | \alpha_d)$, then $\tau(L) = \mathrm{val}(\det M)$ mod d.

(3) Given L, $\exists$ unique $n \in \mathbb{Z}$ s.t.

$\pi^n L \subset L_0$ of the min index

$\tilde{\tau}(L) = \log_q(\text{min index})$

$\tau(L) \equiv \tilde{\tau}(L)$ mod d

(4). 2-neighboring vertices have different colors:

$$\underbrace{\pi L_2 \subset L_1 \subset \underbrace{L_2 \subseteq L_0}_{\text{index } q^{\tau}}}_{\text{index } q^{\tau+d}}$$

index somewhere between $q^{\tau+d}$ and q^{τ}

In particular, a max cell which has dimension $(d-1)$ and hence contains d vertices, has all color, each one exactly once.

Action of $GL_d(F)$

The group $GL_d(F)$ acts on the lattices transitively

$Z(GL_d(F)) = \{tI \mid 0 \neq t \in F\}$ preserves equiv class

easy Exercise: $PGL_d(F)$ acts on the equiv. classes and preserves the combinatorial structure, i.e. edges of the building.

Warning the action of $PGL_d(F)$ does not preserve τ (coloring).

$g \in GL_d(F)$, $L = \mathcal{O}\alpha_1 + \dots + \mathcal{O}\alpha_d$

$M_L = (\alpha_1 | \dots | \alpha_d)$

$\tau(L) = \operatorname{val}\det(M_L) \mod d$

$gL = \mathcal{O}g\alpha_1 + \dots + \mathcal{O}g\alpha_d$

$M_{gL} = (g\alpha_1 | \dots | g\alpha_d) = g \cdot M_L$

$\tau(gL) = \operatorname{val}\det(g \cdot M_L) = \operatorname{val}(\det(g) \cdot \det(M_L)) =$

$= \operatorname{val}(\det g) + \operatorname{val}(\det M_L) \mod d$

i.e. g preserves the colors $\iff \operatorname{val}(\det(g)) \equiv 0 \pmod d$

Denote $G^0 = \{g \mid g \text{ preserves colors}\}$

claim (1) G^0 is a subgr.; it contains $SL_d(F)$

Note $GL_d(F) / SL_d(F) \underset{\det}{\xrightarrow{\simeq}} F^* \simeq \pi^{\mathbb{Z}} \times \mathcal{O}^*$.

(2) $G/G_0 \simeq \mathbb{Z}/d\mathbb{Z}$

Given: $e = ([L_1],[L_2])$ an oriented edge, then $\tau(\vec{e}) =$
$= \tau(L_2) - \tau(L_1) \in \mathbb{Z}/d\mathbb{Z}$

Claim $PGL_d(F)$ preserves the coloring of oriented edges

Link If X is a s.c., $\tau \in X(i)$,

L_τ = link of τ = link of X at τ is the set of all subsets of the form

$$\{\sigma \setminus \tau \mid \tau \subseteq \sigma \in X\}$$

It's a s.c. ~~by~~ itself

Ex $\dim L_\tau = \dim X - \dim \tau - 1$ (assume X is pure)

In particular, if τ is a vertex v, then

$\dim L_v = \dim X - 1$.

Claim If $B = B_d(F)$ and $v \in B(0)$

$L_v = S(d, \mathbb{F}_q)$ - the flag complex of $\mathbb{F}_q^d$

Regarding dimension, it works fine:

$\dim B = d-1$, $\dim S(d, \mathbb{F}_q) = d-2$.

Proof: Note As $PGL_d(F)$ acts transitively on the vertices, the structure of L_v is ind. of v.

Proof Look at L_0: