F - local non-arch field, $\mathcal{O}$ - ring of integers, 10/12/13

m = max ideal, $\mathcal{O}/m\mathcal{O} = \mathbb{F}_q$
$\quad\ \ \pi\mathcal{O}$

$L \le V = F^d$, $L = \sum \mathcal{O}\alpha_i$, $\{\alpha_1, \dots, \alpha_d\}$ - an F-basis of V

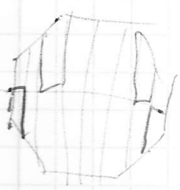
$L_1 \sim L_2 \iff L_1 = \pi^e L_2$ for some $e \in \mathbb{Z}$

$\{[L_0], [L_1], \dots, [L_i]\}$ form an i-cell if $\exists\, L_i' \in [L_i]$,

$\pi L_i' < L_0' < \dots < L_i'$.

$\mathcal{B} = \tilde{A}_{d-1}(F)$
$\ \ \|$
$\mathcal{B}_d(F)$

PGL$_d(F)$ acts on $\mathcal{B}$, trans. on vertices, the standard vertex $L_0 = \sum_{i=1}^{d} \mathcal{O}e_i$



Link at v is $\mathcal{B}_v = \{\sigma \setminus \{v\} \mid \sigma \in \mathcal{B},\ v \in \sigma\}$

$V(\mathcal{B}_v) = \{w \in V(\mathcal{B}) \mid (v, w) \in \mathcal{B}^{(1)}\}$

$\{w_1, \dots, w_i\}$ is an $(i-1)$-cell of $\mathcal{B}_v$ iff $\{v, w_1, \dots, w_i\}$ is an i-cell of $\mathcal{B}$.

$\therefore$ all the neighboring vertices of $[L_0]$ are represented by some L: $\pi L_0 < L \subsetneq L_0$. Any two such L's are not equiv. to each other.

So $\deg([L_0]) = \#\{W \le \mathbb{F}_q^d = L_0/\pi L_0 \mid 0 \ne W \ne \mathbb{F}_q^d\}$
W-subspace

since $L_0 = \sum \mathcal{O}e_i$, $\pi L_0 = \sum \pi\mathcal{O}e_i$ $\Rightarrow L_0/\pi L_0 = \sum (\mathcal{O}/\pi\mathcal{O})e_i \simeq \mathbb{F}_q^d$.

Conclude $\mathcal{B}_{[L_0]} = \mathcal{B}_v \simeq \mathcal{S}(d, q)$.

$\underline{d=2}$ $\mathcal{S}(2,q)$ = a set of $(q+1)$ points.

$d=3$ $\mathcal{S}(3,q)$ = the pts vs lines bipartite graph of the proj plane over $\mathbb{F}_q$.

Cor $B_2 = B_2(F)$ is a $(q+1)$-regular graph

Theorem B_2 is a $(q+1)$-regular **tree**

Pf We need to prove: a) B_2 is connected

b) no cycles

a) any vertex, then $\exists$ rep L' of $[L]$ s.t. $L' \subset L_0$, look at L_0 / L'

we have a sequence of submodules

$$\begin{array}{l} L_0 \\ | \\ L_1 \\ | \\ L_2 \\ \vdots \\ L' \end{array} \quad \text{s.t. } L_{i+1} \supset \pi L_i \text{, i.e. } [L_i] - [L_{i+1}]$$

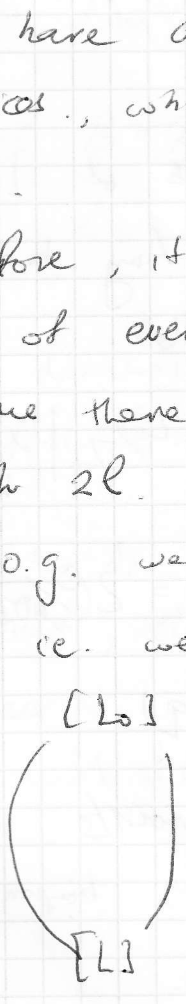
So every vertex is connected to $[L_0]$ and hence B is conn.

b) we have a coloring/types function on the vertices, which makes B into a bi-partite graph.

Therefore, if there is a cycle, it must be of even length.

Assume there is a cycle of minimal length $2l$.

W.l.o.g. we can assume it passes through $[L_0]$, i.e. we have two diff paths of length l from $[L_0]$ to $[L]$ and there's no shorter paths



Let L' be a representative of $[L]$ inside L_0 of minimal possible index in L_0

say $[L_0, L'] = q^{l'}$.

<u>Claim</u> $\operatorname{dist}([L_0],[L]) = l'$.

<u>pf</u>. Indeed (i) $L_0/_{L'}$ is a cyclic module

otherwise $\pi^{-1}(L') \subseteq L_0$, i.e. L' is not a repr. of a minimal index.

we claim that the "clear path from L_0 to L' is the only path of this length.

<u>Claim</u>: Only one min length path from L_0 to L since L' and L'' represent the same vertex so wlog $L' = \pi^e L''$, $e \geq 0$, but now $e \geq 1$, then $L/_{L'}$ is not cyclic, since $L' \subset \pi^e L'' \subset \pi^e L_0$ and $L_0/_{\pi^e L_0}$ is 2-gen (not 1-gen). Contr.,

so $L'' = L'$.

$$\begin{tikzcd}
{[L_0]} \arrow[d, shift left, no head] \arrow[d, shift right, no head] \\
{[L'] = [L'']}
\end{tikzcd}$$

3

<u>Def</u> Let G be a locally compact group with Haar measure μ, i.e. $\mu(gA) = \mu(A)$ $\forall g \in G$, A-measurable.

A <u>discrete</u> subgp. Γ of G ($\exists$ open neigh U of e in G, $\Gamma \cap U = \{e\}$), is called <u>lattice</u>, if $\mu(G/_\Gamma) < \infty$.

Eq. ($\sim$) $\exists$ $F \subseteq G$, $\mu(F) < \infty$ s.t. $\Gamma : F = G$.

This is the case if $G/_\Gamma$ is compact.

In which case Γ is cocompact uniform lattice.

$G = PGL_d(F)$.

Assume Γ is ~~uniform~~ a cocompact lattice in G

$$B = \mathcal{B}_d(F), \quad V(B) = PGL_d(F)/PGL_d(\mathcal{O})$$

$$\Gamma \backslash V(B) = \Gamma \backslash PGL_d(F) / PGL_d(\mathcal{O})$$ — a discrete compact set, ie finite.

e.g. $d=2$, $\Gamma \leq PGL_2(F)$

$\Gamma \backslash B$ has a structure of a graph maybe with „ramification"

If Γ satisfies $\text{dist}(\gamma x, x) \geq 2$, $\forall x \in V(B)$, $\gamma \in G$

then $\Gamma \backslash B$ is a simplicial complex.

This happen if $\Gamma \leq G^+$ = the element of PGL_d preserves τ

$$[PGL_d(F) : G] = d.$$

Every such Γ can be replaced by a subgp Γ' of index $\leq d$ s.t. $\Gamma_1 \backslash B$ is s.c.

An ex. of such lattice in $PGL_2(\mathbb{Q}_p)$

Quaternion algebra

$$H(\mathbb{R}) = \left\{ a + bi + cj + dk \;\middle|\; \begin{array}{l} i^2 = j^2 = k^2 = -1 \\ ij = -ji = k \\ a, b, c, d \in \mathbb{R} \end{array} \right\}$$ 4-dim $\mathbb{R}$-alg

Exercise 1) $H(\mathbb{C}) \simeq M_2(\mathbb{C})$.

2) If $\sqrt{-1} \in R \Rightarrow H(R) = M_2(R)$
(ie. H splits over R)

Example If $p \equiv 1$ (4), then $\sqrt{-1} \in \mathbb{Q}_p$.
(Use Hensel lemma to deduce this from $\sqrt{-1} \in \mathbb{F}_p$).

A more general result $H(\mathbb{Q}_p) \simeq M_2(\mathbb{Q}_p)$ $\forall$ odd prime $p \neq 2$.

For $H(\mathbb{R}) \neq M_2(\mathbb{R})$

Look at $H(R)^* =$ invertible elements of $H(R)$.

$\alpha = a + bi + cj + dk$, denote $\bar{\alpha} = a - bi - cj - dk$.

$\|\alpha\| = a^2 + b^2 + c^2 + d^2$

claim $\alpha \in H(R)$ is invertible $\Leftrightarrow$ $\|\alpha\| \in R^*$

general ring, not reals

pf $\|\alpha\beta\| = \|\alpha\| \cdot \|\beta\|$ (check)

$\Leftarrow$ $\alpha^{-1} = \dfrac{\bar{\alpha}}{\|\alpha\|}$ $\square$

Denote $D(R) = H(R)^* / Z(H(R)^*)$

Ex: $Z(H(R)^*) = R^* \hookrightarrow H(R)^*$

$a \longrightarrow a + 0 \cdot i + a \cdot j + 0 \cdot k$

Now If H splits over a field, e.g. $\mathbb{C}$, $\mathbb{Q}_p$, $p \neq 2$

$$D(F) = (M_2(F))^*/Z = GL_2(F)/Z = PGL_2(F)$$

$F = \mathbb{R}$ $D(\mathbb{R}) = H(\mathbb{R})^*/Z \overset{\text{Exercise}}{\simeq} SO(3, \mathbb{R})$

$$H(\mathbb{R})^* = \{\alpha \in H(\mathbb{R}) \mid \|\alpha\| \neq 0\} = \mathbb{R}^4 \setminus \{0\}.$$

$H(\mathbb{R})^*/\mathbb{R}^* = P_3(\mathbb{R}) =$ proj space which is compact.

Take $R = \mathbb{Z}[\frac{1}{p}] \hookrightarrow \mathbb{R}$ (dense subring)

$\hookrightarrow \mathbb{Q}_p$ (dense subring)

$\hookrightarrow \mathbb{R} \times \mathbb{Q}_p$ (discrete subring !!!)

diagonal embedding

Therefore $\Gamma = H(\mathbb{Z}[\frac{1}{p}])^{(*)} \hookrightarrow H(\mathbb{R})^{(*)} \times H(\mathbb{Q}_p)^{(*)}$ discrete.

$(/Z)$ $(/Z)$ $(/Z)$

$\Gamma \hookrightarrow SO(3) \times PGL_2(\mathbb{Q}_p)$

Exercise $\Gamma \hookrightarrow PGL_2(\mathbb{Q}_p)$ is discrete since

$SO(3)$ is compact.

Next time: this is a lattice.

24.12
Alex

R – commun ring with Id

$H(R) = \{x_0 + x_1 i + x_2 j + x_3 k \mid x_i \in R,\ i^2 = j^2 = k^2 = -1,\ ij = k = -ji\}$

If K is a field with $\varepsilon = \sqrt{-1} \in K$, then

$H(K) \simeq M_2(K)$

$$\alpha = x_0 + x_1 i + x_2 j + x_3 k \xmapsto{\eta} \begin{pmatrix} x_0 + \varepsilon x_1 & x_2 + \varepsilon x_3 \\ -x_2 + \varepsilon x_3 & x_0 - \varepsilon x_1 \end{pmatrix}$$

Note $\bar{\alpha} = x_0 - x_1 i - x_2 j - x_3 k$,

$$\alpha \bar{\alpha} = \|\alpha\| = \sum_{i=0}^{3} x_i^2$$

Note $\det(\eta(\alpha)) = \|\alpha\|$

Say H splits over K if $H(K) \simeq M_2(F)$

So if $\sqrt{-1} \in K$ then H splits but not iff

In fact for every $p \neq 2$, H splits over $\mathbb{Q}_p$

(but only if $p \equiv 1\ (4)$, $\sqrt{-1} \in \mathbb{Z}_p \subset \mathbb{Q}_p$)

H does not split over $\mathbb{R}$, so $H(\mathbb{R})$ is a division algebra, $H^*(\mathbb{R}) = H(\mathbb{R}) \setminus \{0\}$,

$$H(\mathbb{R})^*/Z \simeq SO(3, \mathbb{R}), \text{ ie compact}$$

$Z \leftarrow \mathbb{R}^*$

Let p be odd (we will work with $p \equiv 1\ (4)$)

$$H(\mathbb{Z}[\tfrac{1}{p}]) \xhookrightarrow[\text{discrete}]{} H(\mathbb{R}) \times H(\mathbb{Q}_p)^*/Z$$